



The Open
University

MST125

Essential mathematics 2

Exercise Booklet 10

1 Motion along a straight line

Exercise 1

A particle travels along a straight line. Its position x at time t is given by

$$x = \frac{1}{2}t^3 - 2t.$$

Find the following in terms of the time t .

- The velocity of the particle.
- The acceleration of the particle.

Exercise 2

A car accelerates from rest. Its acceleration a (in m s^{-2}) as it pulls away is modelled by the equation

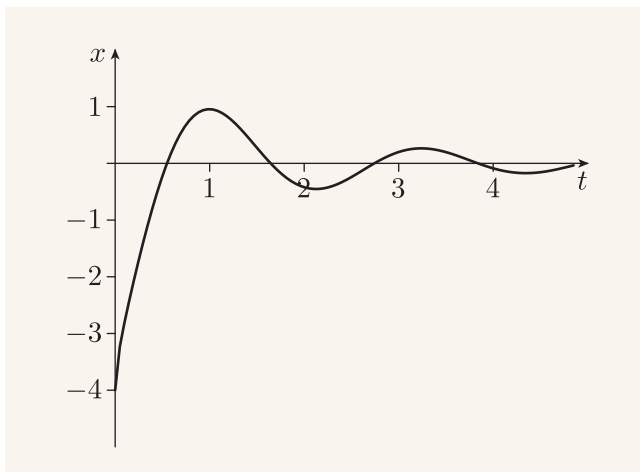
$$a = t + e^{\frac{1}{2}t},$$

where t (in seconds) is the time since the car began to accelerate.

- Find the velocity v (in m s^{-1}) of the car in terms of the time t .
- Find the position x (in m) of the car in terms of the time t .
- How far has the car travelled after 6 seconds? Give your answer to three significant figures.

Exercise 3

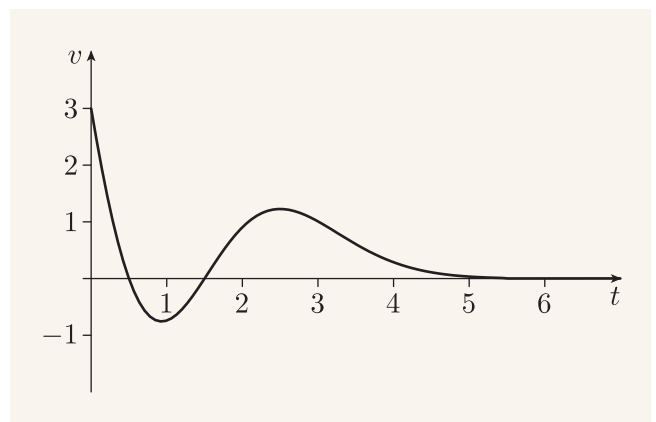
The graph below shows the position (in metres) of a particle against time (in seconds).



- How many times does the direction of motion of the particle reverse?
- What is the approximate speed of the particle after one second?
- What is the approximate total distance that the particle has travelled after two seconds?
- At approximately what time is the speed of the particle greatest?

Exercise 4

The graph below shows the velocity (in m s^{-1}) of a particle against time (in seconds).



- How many times does the direction of motion of the particle reverse?
- At which of the two times $t = \frac{1}{2}$ and $t = \frac{3}{2}$ is the particle further from its position at time $t = 0$?
- At approximately what time is the acceleration of the particle greatest?
- Is the acceleration of the particle ever zero? If so, when?

Exercise 5

A lorry pulls away from rest with a constant acceleration of 2 m s^{-2} .

- How long does it take for the lorry to reach a speed of 13 m s^{-1} ?
- How far does it travel in this time?

Exercise 6

A peregrine falcon, initially hovering (stationary) in the air, is subsequently observed flying towards some prey. It takes the falcon 5 seconds to reach its prey, and it is estimated that the distance that it travels in this time is 220 m. Making the assumption that the acceleration of the falcon is constant, calculate the following.

- (a) The acceleration of the falcon.
 - (b) The speed of the falcon when it reaches its prey.
-

Exercise 7

A car travelling at 60 mph (26.8 m s^{-1}) in a 60 mph zone approaches a sign for 30 mph. The sign is 100 metres away. The driver wishes to have slowed down to 30 mph (13.4 m s^{-1}) by the time he reaches the sign. In the following, assume that the deceleration of the car is constant.

- (a) What should the rate of the car's deceleration be in order to achieve this?
- (b) How long will it take?

Give your answers to three significant figures.

Exercise 8

To determine the height of a bridge, someone drops a small stone (from rest) from the bridge so that it falls vertically downwards under the influence of gravity alone, until it hits the ground below. It takes the stone 3.5 seconds to reach the ground.

How high is the bridge? Give your answer in metres to two significant figures.

Exercise 9

A rocket is fired vertically into the air. After being launched, it rises until its motor cuts out. At this time, the rocket has attained a height of 700 metres above the launch pad, and a velocity of 63 m s^{-1} upwards. From this time on, the rocket is acted on by gravity alone, so it continues to rise to a maximum height and then crashes back onto the launch pad.

Take the origin to be the point where the rocket's motor cuts out, and measure time from the moment when the motor cuts out.

- (a) How far does the rocket travel from the moment when its motor cuts out to the time when it reaches its maximum height? Give your answer in metres to two significant figures.
 - (b) How long after the motor cuts out does the rocket crash back onto the launch pad? Give your answer in seconds to two significant figures.
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Exercise 10

The Shanghai MagLev train travels a distance of 30.5 km between Pudong Airport and Longyang Road Station. The train leaves Pudong Airport with a constant acceleration of 0.57 m s^{-2} until it reaches a speed of 119.7 m s^{-1} (431 km h^{-1}). It then travels at this speed for 60 seconds, before slowing down with constant deceleration to stop at Longyang Road Station.

Give the following answers to three significant figures. (Don't forget to use the unrounded values in subsequent calculations.)

- (a) How far does the MagLev train travel while its speed is increasing?
 - (b) How far does the train travel at its top speed?
 - (c) What must the rate of deceleration be in the final stage of the journey to ensure that the train comes to a stop at the station?
 - (d) How long does it take the train to travel from Pudong Airport to Longyang Road Station?
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Exercise 11

While making an emergency stop, a car slows with acceleration given by $-\frac{2}{9}x^2$ (in m s^{-2}), where x is the distance (in metres) from where the driver starts applying the brakes. The car is travelling with a speed of 16 m s^{-1} when the driver begins braking.

Find the following.

- The speed of the car after it has travelled 9 metres, to the nearest m s^{-1} .
- The distance travelled by the car from where the driver begins braking to where it comes to a stop.

2 Newton's second law of motion

Exercise 12

A ball of mass 0.4 kg accelerates from rest in a straight line along a sloping seesaw whose tilt does not change. The distance x (in metres) of the ball from its starting point is given in terms of the time t (in seconds) by

$$x = t^2 + \frac{1}{5}(1 - e^t).$$

Find the resultant force acting on the ball in terms of the time t .

Exercise 13

An object of mass 3 kg hanging on the end of a spring is released from rest, 1 metre above a fixed point, and bounces vertically up and down. Take the x -axis to lie along the line of motion of the object, with the origin at the fixed point and the positive direction pointing upwards. The resultant force F (in newtons) acting on the object is given by

$$F = -12 \cos 2t,$$

where t is the time since the object was released.

Find the position of the object in terms of the time t .

Exercise 14

An asteroid of mass $20\,000 \text{ kg}$ heads for impact with a planet, moving along a straight line towards the centre of the planet. Take the origin to be the centre of the planet, with the x -axis pointing in the direction from the centre of the planet to the asteroid, so that x is the displacement (in metres) of the asteroid from the planet. The resultant force F (in newtons) acting on the asteroid is given by

$$F = -\frac{2.4 \times 10^{19}}{x^2}.$$

When the asteroid is first observed, it is $25\,000 \text{ km}$ from the planet and travelling at a speed of 4 km s^{-1} towards the planet.

The planet has a radius of 5000 km . Find the speed of the asteroid on impact with the planet.

(Don't forget to write positions and velocities in terms of metres, to be consistent with the force being given in newtons.)

Exercise 15

A sled of mass 40 kg is pulled from rest along a smooth horizontal surface with a rope that makes an angle of 50° with the horizontal. The magnitude of the pulling force is 72 N .

Find the magnitude of the acceleration of the sled, and its speed after 5 seconds. Give your answers to two significant figures.

Exercise 16

A child of mass 21 kg slides down from the top of a smooth slide, from rest, under the effect of gravity alone. The slide is inclined at an angle of 35° to the horizontal and is 3 metres long.

How long does it take the child to slide down the slide? Give your answer to two significant figures.

Exercise 17

A curling stone is given a push so that it slides along the ice in a straight line, with an initial speed of 5 m s^{-1} . The coefficient of sliding friction between the stone and the ice is 0.05.

The person who pushes the curling stone is aiming to hit another stone 30 metres away. If the person aims in the correct direction, will this be achieved?

Exercise 18

A box is given an initial push up an inclined plane so that it slides upwards for a short time before coming to a stop. The plane is inclined at an angle of 30° to the horizontal, and the coefficient of sliding friction between the box and the plane is 0.4.

Find, to two significant figures, what the initial speed of the box must be so that it travels a distance of one metre up the slope.

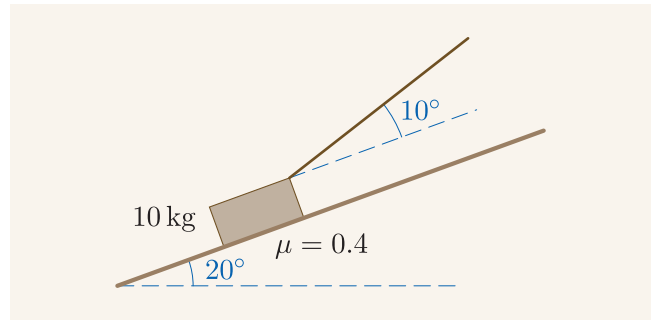
Exercise 19

A car is travelling down an icy ramp when its brakes lock and it begins to slide. The ramp is inclined at an angle of 15° to the horizontal, and the coefficient of sliding friction between the car and the ramp is 0.35. When the brakes lock, the car is travelling at 11 m s^{-1} . The car slides for 5 seconds before the driver regains control of the car.

What is the speed of the car when the driver regains control? Give your answer to two significant figures.

Exercise 20

A box of mass 10 kg is pulled up a rough slope, as shown in the diagram below. The slope is inclined at an angle of 20° to the horizontal. The box is pulled by a rope inclined at an angle of 10° to the slope, with a force of magnitude 80 N. The coefficient of sliding friction between the slope and the box is 0.4.



Calculate the acceleration of the box up the slope, in m s^{-2} to two significant figures.

3 Motion in two or three dimensions

Exercise 21

The position of a particle is given in terms of the time t by

$$\mathbf{r} = 0.5t^3 \mathbf{i} + (2t + 4) \mathbf{j} - 0.25t^2 \mathbf{k}.$$

Find the position and the distance from the origin of the particle at each of the following times.

- (a) $t = 0$ (b) $t = 2$ (c) $t = 10$

In part (c), give your answer for the distance to two decimal places.

Exercise 22

The position of a particle is given in terms of the time t by

$$\mathbf{r} = 0.5t^3 \mathbf{i} + (2t + 4) \mathbf{j} - 0.25t^2 \mathbf{k}.$$

Find the following.

- (a) The velocity of the particle in terms of the time t .
 (b) The speed of the particle when $t = 10$.
 Give your answer to two decimal places.

Exercise 23

The velocity $\mathbf{v}$ of a particle is given in terms of the time t by

$$\mathbf{v} = 2e^{-t}\mathbf{i} + 2t^3\mathbf{j} - t\mathbf{k}.$$

At time $t = 0$, the position of the particle is $\mathbf{r} = \frac{1}{2}\mathbf{j}$.

Find the position and the distance from the origin of the particle at time $t = 4$. Give your answer for the distance to two decimal places.

Exercise 24

A child slides down a helter skelter ride. The position $\mathbf{r}$ (in metres) of the child is given in terms of the time t (in seconds) by

$$\mathbf{r} = 2 \cos \frac{2\pi t}{5} \mathbf{i} + 2 \sin \frac{2\pi t}{5} \mathbf{j} + \left(12 - \frac{4t}{5}\right) \mathbf{k}.$$

Find the following.

- The velocity of the child in terms of t .
- The speed of the child when $t = 5$, in m s^{-1} to three significant figures.
- The acceleration of the child in terms of t .

Exercise 25

The acceleration (in m s^{-2}) of a skier as she slaloms down a hill is given in terms of t , the time (in seconds) after she began her descent, by

$$\mathbf{a} = \frac{1}{2}\mathbf{i} - \pi^2 \sin \pi t \mathbf{j}.$$

The initial position of the skier at time $t = 0$ is $\mathbf{r} = 4\mathbf{k}$, and her initial velocity is $\mathbf{v} = \pi\mathbf{j} - \frac{1}{2}\mathbf{k}$.

- Find the velocity of the skier in terms of the time t .
- Find the position of the skier in terms of the time t .

Exercise 26

A pebble is thrown from the edge of a cliff into the sea. It is thrown at an angle of 30° above the horizontal, with speed 6 m s^{-1} . The edge of the cliff is 32 metres above the sea.

- What is the maximum height above the sea reached by the pebble?
- How long does it take before the pebble splashes into the sea?

Give your answers to two significant figures.

Exercise 27

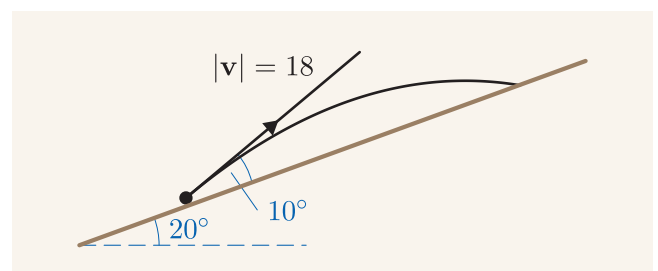
An archer is trying to shoot an arrow towards a target that is 90 metres away. The archer will release the arrow from the same height as the centre of the target, with a speed of 42 m s^{-1} .

At what angle above the horizontal should the archer point the arrow to ensure that it hits the centre of the target?

Hint: the identity $\sin 2\theta = 2 \sin \theta \cos \theta$ may prove useful.

Exercise 28

A ball is kicked up a slope that is inclined at an angle of 20° to the horizontal. The ball is kicked with an initial speed of 18 m s^{-1} , at an angle of 10° to the slope, as shown in the diagram below.

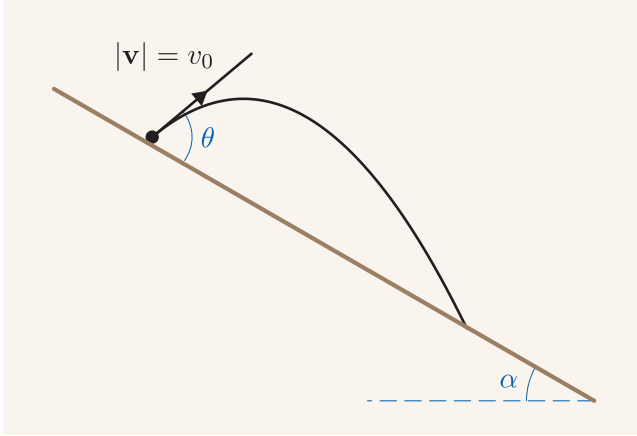


Calculate how far up the slope the ball lands, in metres to two significant figures.

You should take the x -axis to be parallel to the slope, to make the working easier.

Exercise 29

A slope is inclined at an angle α to the horizontal. A ball is thrown down the slope with initial speed v_0 , at an angle θ to the slope, as shown in the diagram below.



- (a) Show that the distance that the ball has travelled down the slope when it lands is given by

$$\frac{2v_0^2 \sin \theta (\cos \theta \cos \alpha + \sin \theta \sin \alpha)}{g \cos^2 \alpha},$$

where g is the magnitude of the acceleration due to gravity.

Explain why we can write this expression as

$$\frac{2v_0^2 \sin \theta \cos(\theta - \alpha)}{g \cos^2 \alpha}.$$

- (b) Show that if $\alpha = 0$, then this expression reduces to

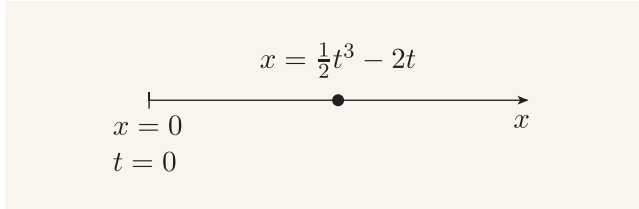
$$\frac{v_0^2 \sin 2\theta}{g},$$

as in the solution to Activity 28 in Unit 10.

Solutions to exercises

Solution to Exercise 1

A diagram of the situation is shown below.



- (a) The velocity v is found by differentiating the position x . This gives

$$v = \frac{dx}{dt} = \frac{d}{dt} \left(\frac{1}{2}t^3 - 2t \right) = \frac{3}{2}t^2 - 2.$$

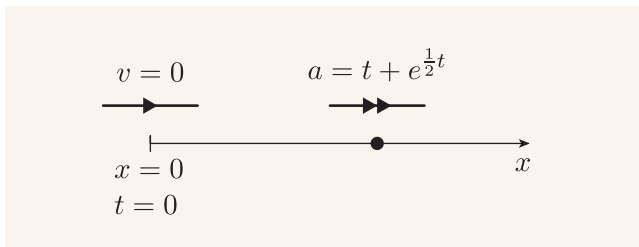
- (b) The acceleration a is found by differentiating the velocity v . This gives

$$a = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{3}{2}t^2 - 2 \right) = 3t.$$

Solution to Exercise 2

Model the car as a particle.

A diagram of the situation is shown below. We are told that the car starts from rest, so $v = 0$ when $t = 0$.



- (a) To find the velocity, we integrate the acceleration with respect to t :

$$\begin{aligned} v &= \int a \, dt \\ &= \int (t + e^{\frac{1}{2}t}) \, dt \\ &= \frac{1}{2}t^2 + 2e^{\frac{1}{2}t} + c, \end{aligned}$$

where c is a constant.

Using the initial condition that $v = 0$ when $t = 0$ gives

$$0 = \frac{1}{2} \times 0^2 + 2e^{\frac{1}{2} \times 0} + c,$$

so $c = -2$.

The velocity of the car, in m s^{-1} , in terms of the time t is therefore given by

$$v = \frac{1}{2}t^2 + 2e^{\frac{1}{2}t} - 2.$$

- (b) To find the position, we integrate the velocity with respect to t :

$$\begin{aligned} x &= \int v \, dt \\ &= \int \left(\frac{1}{2}t^2 + 2e^{\frac{1}{2}t} - 2 \right) \, dt \\ &= \frac{1}{6}t^3 + 4e^{\frac{1}{2}t} - 2t + d, \end{aligned}$$

where d is a constant.

Using the initial condition that $x = 0$ when $t = 0$ gives

$$0 = \frac{1}{6} \times 0^3 + 4e^{\frac{1}{2} \times 0} - 2 \times 0 + d,$$

so $d = -4$.

The position of the car, in metres, in terms of the time t is therefore given by

$$x = \frac{1}{6}t^3 + 4e^{\frac{1}{2}t} - 2t - 4.$$

- (c) To find the position after 6 seconds, we substitute $t = 6$ into the expression for the position found in part (b). This gives

$$\begin{aligned} x &= \frac{1}{6} \times 6^3 + 4e^{\frac{1}{2} \times 6} - 2 \times 6 - 4 \\ &= 100.342147 \dots \end{aligned}$$

That is, the distance travelled after 6 seconds is 100 m, to three significant figures.

Solution to Exercise 3

- (a) The direction of the graph changes four times (there are four turning points), so the direction of motion of the particle reverses four times.
- (b) After one second, at time $t = 1$, the graph appears to have a turning point, so the speed of the particle at this time is approximately 0 m s^{-1} .
- (c) The graph shows that the particle starts at position $x = -4$, and that at time $t = 1$ its position is approximately $x = 1$, so the particle has travelled approximately 5 metres after one second.

At this time, the particle changes direction and, from being at position $x = 1$ at time $t = 1$, it then reaches position $x = -\frac{1}{2}$, approximately, at time $t = 2$. So the particle has travelled approximately 1.5 metres between time $t = 1$ and time $t = 2$.

Therefore the total distance travelled by the particle after two seconds is approximately $5 + 1.5 = 6.5$ metres.

- (d) The speed of the particle is greatest when the magnitude of the gradient of the position–time graph is greatest. This is at time approximately 0 seconds, just after the motion begins.

Solution to Exercise 4

- (a) When the velocity is positive, the particle is moving in the positive x -direction, and when the velocity is negative, the particle is moving in the negative x -direction. So the direction of motion of the particle reverses when the curve crosses the t -axis.

The curve crosses the t -axis twice, so the direction of motion reverses twice.

- (b) The change in the position of the particle between any two times is the signed area between the velocity–time graph and the t -axis between these two times.

Between time $t = 0$ and time $t = \frac{1}{2}$, the area between the graph and the t -axis lies entirely above the t -axis, so the signed area is positive and hence at time $t = \frac{1}{2}$ the particle has a positive displacement from its position at time $t = 0$.

Between time $t = 0$ and time $t = \frac{3}{2}$, the signed area also includes an area below the t -axis (just slightly smaller than the area above), which makes a negative contribution to the signed area. This will cancel out much of the positive signed area, and hence the total signed area between time $t = 0$ and time $t = \frac{3}{2}$ is much smaller in magnitude than that between time $t = 0$ and time $t = \frac{1}{2}$.

Hence the particle is further from its position at time $t = 0$ when $t = \frac{1}{2}$ than when $t = \frac{3}{2}$.

- (c) The acceleration of the particle is greatest when the magnitude of the gradient of the velocity–time graph is greatest. This is at time approximately 0 seconds, just after the motion begins.

- (d) Yes. The acceleration of the particle is the gradient of the velocity–time graph. This is zero at the two turning points, at times $t = 0.9$ and $t = 2.5$, approximately. It is also very close to zero from time $t = 5$ to time $t = 7$, since the graph is very close to being horizontal in this time interval.

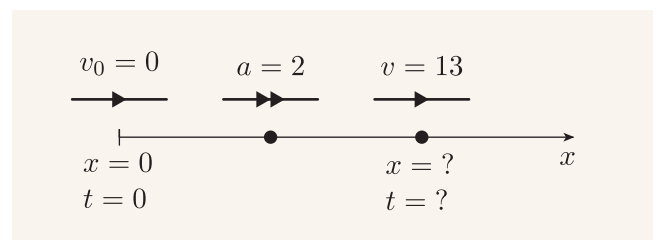
Solution to Exercise 5

Model the lorry as a particle.

Measure time from the moment when the lorry starts accelerating.

Choose the positive direction of the x -axis to be the direction in which the lorry is accelerating, with the origin at the point where the lorry is at time $t = 0$.

The position of the lorry at time t is x . Since the lorry starts from rest, its initial velocity is $v_0 = 0$. Its acceleration is $a = 2$.



- (a) We know that $v_0 = 0$ and $a = 2$, and we want to find t when $v = 13$. Using $v = v_0 + at$ gives

$$13 = 0 + 2t,$$

so $t = 6.5$.

The time that it takes for the lorry to reach 13 m s^{-1} is 6.5 seconds.

- (b) Here we again know that $v_0 = 0$ and $a = 2$, but this time we want to find x when $v = 13$. Using $v^2 = v_0^2 + 2ax$ gives

$$13^2 = 0^2 + 2 \times 2 \times x,$$

so

$$x = \frac{13^2}{2 \times 2} = 42.25.$$

The distance travelled in this time is 42.25 metres.

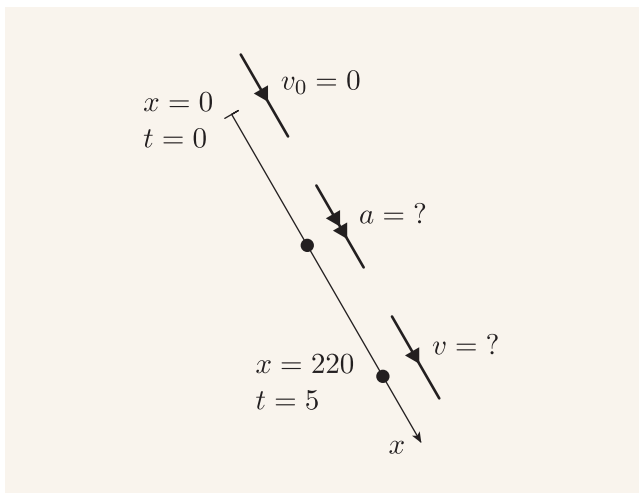
Solution to Exercise 6

Model the falcon as a particle.

Measure time from the moment when the falcon begins flying towards its prey.

Choose the positive direction of the x -axis to be the direction in which the falcon is accelerating, with the origin at the point where the falcon is at time $t = 0$.

The position of the falcon at time t is x . Since the falcon is initially hovering, we can take its initial velocity to be $v_0 = 0$. The distance travelled is $x = 220$, and the time taken is $t = 5$. We assume that the acceleration is constant.



- (a) We know that $v_0 = 0$, $x = 220$ and $t = 5$. We want to calculate the value of a .

Using $x = v_0 t + \frac{1}{2} a t^2$ gives

$$220 = 0 \times 5 + \frac{1}{2} a \times 5^2.$$

Hence

$$a = \frac{2 \times 220}{5^2} = 17.6.$$

So the acceleration of the falcon is 17.6 m s^{-2} .

- (b) Here we again know that $v_0 = 0$, $x = 220$ and $t = 5$. This time we want to calculate the final velocity v . Using $x = \frac{1}{2}(v_0 + v)t$ gives

$$220 = \frac{1}{2}(0 + v) \times 5,$$

so

$$v = \frac{220 \times 2}{5} = 88.$$

Hence the speed of the falcon when it reaches its prey is 88 m s^{-1} .

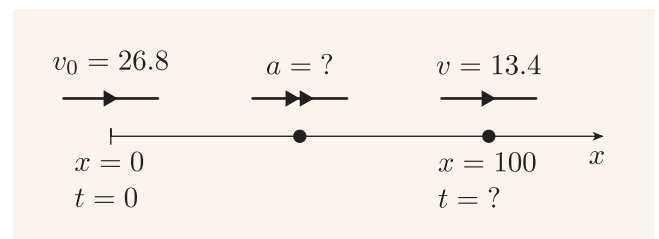
Solution to Exercise 7

Model the car as a particle.

Measure time from the moment when the car starts decelerating.

Choose the positive direction of the x -axis to be the direction in which the car is travelling (so the acceleration is negative), with the origin at the point where the car is at time $t = 0$.

The position of the car at time t is x . The car's initial velocity is $v_0 = 26.8$. The desired final velocity is $v = 13.4$. The car travels distance $x = 100$.



- (a) We know that $v_0 = 26.8$, $v = 13.4$ and $x = 100$. We want to find a . Using $v^2 = v_0^2 + 2ax$ gives

$$13.4^2 = 26.8^2 + 2 \times a \times 100,$$

so

$$a = \frac{13.4^2 - 26.8^2}{2 \times 100} = -2.6934.$$

Hence the driver should decelerate at 2.69 m s^{-2} (to 3 s.f.).

- (b) Here we again know that $v_0 = 26.8$, $v = 13.4$ and $x = 100$, and we want to find t . Using $x = \frac{1}{2}(v_0 + v)t$ gives

$$100 = \frac{1}{2}(26.8 + 13.4)t,$$

so

$$t = \frac{100 \times 2}{26.8 + 13.4} = 4.975124 \dots$$

Hence it will take the driver 4.98 seconds (to 3 s.f.) to slow down to 30 mph.

Solution to Exercise 8

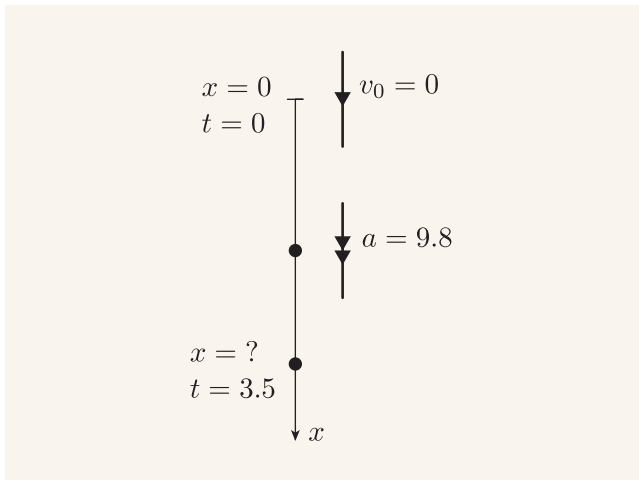
Model the stone as a particle.

Measure time from the moment when the stone is released.

Choose the positive direction of the x -axis to be downwards, with the origin at the point where the stone is released.

The position of the stone at time t is x . The initial velocity of the stone is $v_0 = 0$.

Its acceleration a is the acceleration due to gravity, which has magnitude 9.8 m s^{-2} and is directed downwards, so $a = 9.8$. The time taken is $t = 3.5$.



We know values for v_0 , a and t , and we want to calculate x . Using $x = v_0 t + \frac{1}{2} a t^2$ gives

$$x = 0 \times 3.5 + \frac{1}{2} \times 9.8 \times 3.5^2 = 60.025.$$

Hence the bridge is 60 m high (to 2 s.f.).

Solution to Exercise 9

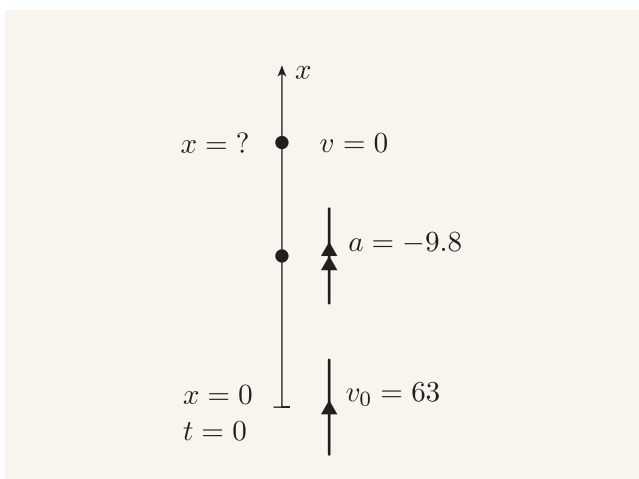
Model the rocket as a particle.

As specified, measure time from the moment when the rocket's motor cuts out.

Choose the positive direction of the x -axis to be upwards, with the origin at the point where the rocket is when its motor cuts out.

The position of the rocket at time t is x . Its initial velocity is $v_0 = 63$. Its acceleration a is the acceleration due to gravity, which has magnitude 9.8 m s^{-2} and is directed downwards, so $a = -9.8$.

(a)



The rocket reaches its maximum height when $v = 0$. So we know that $v_0 = 63$ and $a = -9.8$, and we are seeking the value of x when $v = 0$. Using $v^2 = v_0^2 + 2ax$, we get

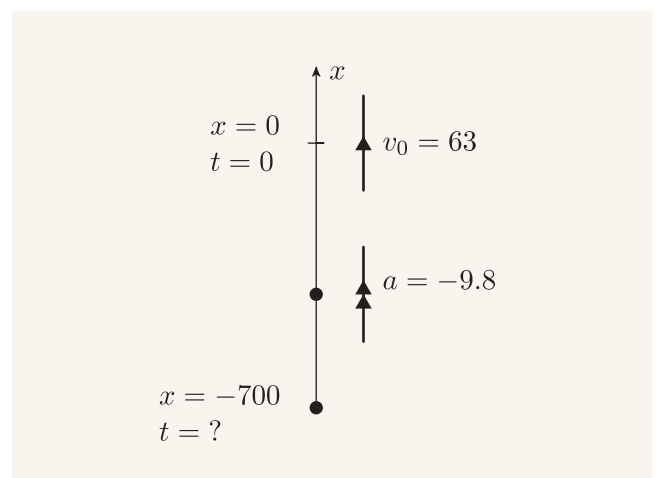
$$0^2 = 63^2 + 2(-9.8)x$$

so

$$x = \frac{-63^2}{-2 \times 9.8} = 202.5.$$

The rocket travels 200 metres (to 2 s.f.) between its motor cutting out and its maximum height.

(b)



The rocket will crash back onto the launch pad when $x = -700$. So we know that $v_0 = 63$ and $a = -9.8$, and we are seeking the time t when $x = -700$. Using $x = v_0 t + \frac{1}{2} a t^2$ gives

$$-700 = 63t + \frac{1}{2}(-9.8)t^2.$$

We can rearrange this equation to give

$$4.9t^2 - 63t - 700 = 0.$$

Multiplying through by $\frac{10}{7}$ gives

$$7t^2 - 90t - 1000 = 0.$$

Factorising gives

$$(7t + 50)(t - 20) = 0,$$

which has solutions $t = -\frac{50}{7}$ and $t = 20$.

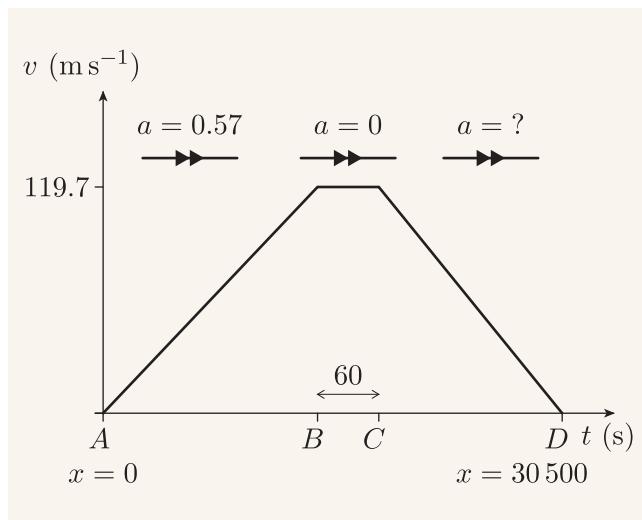
Since the time when the rocket crashes back onto the launch pad is positive, we conclude that this happens 20 seconds (to 2 s.f.) after the motor cuts out.

Solution to Exercise 10

Model the MagLev train as a particle.

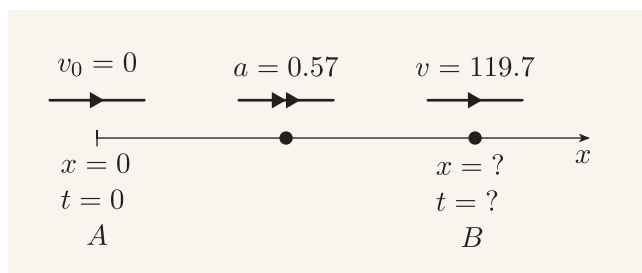
Let A be Pudong Airport, let D be Longyang Road Station, and let B and C be the points in the train's journey where its acceleration changes.

A sketch of a velocity–time graph for the journey is shown below.



We know the accelerations for the first two parts of the journey, and the maximum velocity of the train. We also know that the total distance travelled is 30 500 m.

- (a) For this stage, measure time from when the MagLev train is at point A , and take the x -axis to point in the direction of motion, with the origin at A .



We have that $v_0 = 0$ and $a = 0.57$, and we want to find x when $v = 119.7$. Using $v^2 = v_0^2 + 2ax$, we get

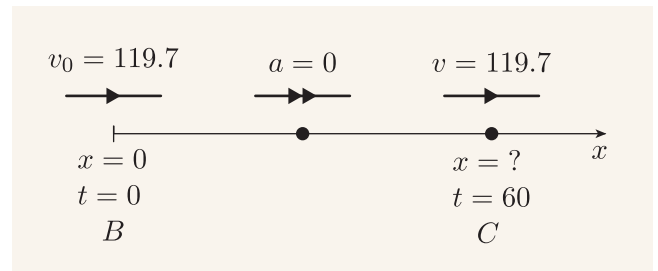
$$119.7^2 = 0^2 + 2 \times 0.57 \times x,$$

so

$$x = \frac{119.7^2}{2 \times 0.57} = 12\,568.5.$$

Hence the distance that the train travels while its speed is increasing is 12.6 km (to 3 s.f.).

- (b) For this stage, measure time from when the MagLev train is at point B , and take the x -axis to point in the direction of motion, with the origin at B .



We have that $v_0 = 119.7$ and $a = 0$, and we want to find x when $t = 60$. Using $x = v_0t - \frac{1}{2}at^2$ (which, with $a = 0$, reduces to $x = v_0t$) gives

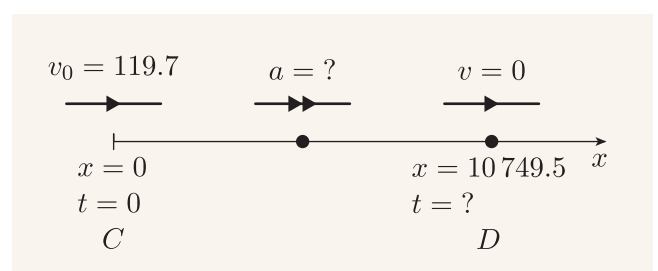
$$x = 119.7 \times 60 = 7182.$$

Hence the distance travelled by the MagLev train while at its top speed is 7.18 km (to 3 s.f.).

- (c) For this stage, measure time from when the MagLev train is at point C , and take the x -axis to point in the direction of motion, with the origin at C .

We know that the total length of the journey is 30 500 metres, and we have calculated the lengths of the first two parts of the journey in parts (a) and (b), so we can find the length x of the final part of the journey:

$$x = 30\,500 - 12\,568.5 - 7182 = 10\,749.5.$$



We have that $v_0 = 119.7$ and $v = 0$. We can therefore use $v^2 = v_0^2 + 2ax$ to calculate a . We obtain

$$0^2 = 119.7^2 + 2 \times a \times 10\,749.5,$$

so

$$a = \frac{-119.7^2}{2 \times 10\,749.5} = -0.666\,45\dots$$

Hence the rate of deceleration must be 0.666 m s^{-2} (to 3 s.f.).

- (d) To calculate the total time taken for the journey, we need to calculate the times taken for stages AB and CD , and add them to the 60 seconds taken for stage BC .

For stage AB , measure time from when the MagLev train is at point A , and take the x -axis to point in the direction of motion, with the origin at A .

We have that $v_0 = 0$, $v = 119.7$ and $a = 0.57$. Using $v = v_0 + at$ gives

$$119.7 = 0 + 0.57 \times t,$$

so

$$t = \frac{119.7}{0.57} = 210.$$

For stage CD , measure time from when the MagLev train is at point C , and take the x -axis to point in the direction of motion, with the origin at C .

We have that $v_0 = 119.7$, $v = 0$ and $x = 10\,749.5$. Using $x = \frac{1}{2}(v_0 + v)t$ gives

$$10\,749.5 = \frac{1}{2}(0 + 119.7)t,$$

so

$$t = \frac{2 \times 10\,749.5}{119.7} = 179.6 \dots$$

Adding the three times together, we get

$$210 + 60 + 179.6 \dots = 449.6 \dots$$

Hence the total time taken for the journey is 450 seconds (to 3 s.f.), that is, 7 minutes and 30 seconds.

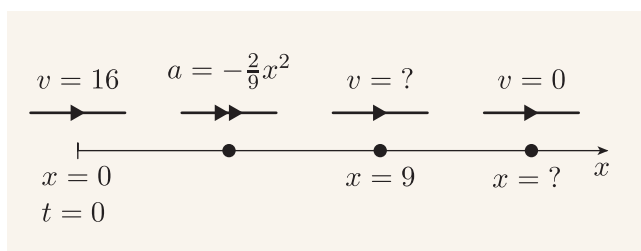
Solution to Exercise 11

Model the car as a particle.

The x -axis points in the direction of travel, with the origin at the point where the driver begins braking.

We know that the velocity is $v = 16$ when the position is $x = 0$.

The acceleration is given by $a = -\frac{2}{9}x^2$.



Using the equation

$$a = v \frac{dv}{dx}$$

gives

$$-\frac{2}{9}x^2 = v \frac{dv}{dx},$$

that is,

$$v \frac{dv}{dx} = -\frac{2}{9}x^2.$$

Separating the variables gives

$$\int v \, dv = - \int \frac{2}{9}x^2 \, dx.$$

Performing the integrations, we get

$$\frac{1}{2}v^2 = -\frac{2}{27}x^3 + c,$$

where c is an unknown constant.

Since we know that $v = 16$ when $x = 0$, we have

$$\frac{1}{2} \times 16^2 = -\frac{2}{27} \times 0^3 + c,$$

so

$$c = \frac{1}{2} \times 16^2 = 128.$$

Hence the velocity is given by

$$\frac{1}{2}v^2 = -\frac{2}{27}x^3 + 128,$$

which we can rearrange to give

$$v^2 = -\frac{4}{27}x^3 + 256.$$

Since the car is moving in the positive x -direction, its velocity v is positive for all values of x involved. So we obtain

$$v = \sqrt{256 - \frac{4}{27}x^3}.$$

- (a) We can use the expression above for the velocity to calculate the velocity when $x = 9$. This gives

$$\begin{aligned} v &= \sqrt{256 - \frac{4}{27} \times 9^3} \\ &= 2\sqrt{37} \\ &= 12.1655 \dots \end{aligned}$$

Hence the speed after 9 metres of braking is 12 m s^{-1} (to the nearest m s^{-1}).

Exercise Booklet 10

- (b) We can use the expression above for the velocity in terms of the position to calculate the position x when $v = 0$. This gives

$$0 = \sqrt{256 - \frac{4}{27}x^3},$$

so

$$0 = 256 - \frac{4}{27}x^3,$$

thus

$$\frac{4}{27}x^3 = 256,$$

hence

$$x^3 = 1728,$$

so

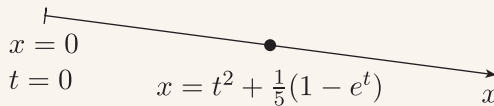
$$x = 12.$$

The car comes to a stop 12 metres from where the driver begins braking.

Solution to Exercise 12

Model the ball as a particle.

The positive direction of the x -axis is along the seesaw, with the origin at the starting point of the ball.



$$x = t^2 + \frac{1}{5}(1 - e^t)$$

The acceleration (in m s^{-2}) of the ball is given by

$$\begin{aligned} a &= \frac{d^2x}{dt^2} = \frac{d}{dt} \left(2t + \frac{1}{5}(1 - e^t) \right) \\ &= \frac{d}{dt} \left(2t - \frac{1}{5}e^t \right) \\ &= 2 - \frac{1}{5}e^t. \end{aligned}$$

We know that the mass of the ball is 0.4 kg, so $m = 0.4$. We can therefore apply Newton's second law, $F = ma$, to get

$$F = ma = 0.4 \times \left(2 - \frac{1}{5}e^t \right) = \frac{2}{25}(10 - e^t).$$

That is, the resultant force F (in newtons) acting on the ball is given in terms of the time t by

$$F = \frac{2}{25}(10 - e^t).$$

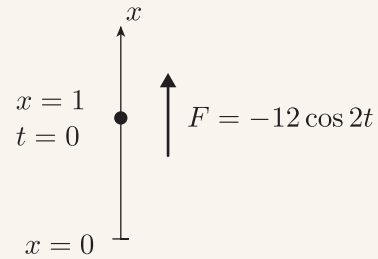
(The direction of the force is the direction of motion of the ball.)

Solution to Exercise 13

Model the object as a particle.

As indicated in the question, take the x -axis to lie along the line of motion, with its positive direction pointing upwards, and the origin at the location of the object at time $t = 0$.

Then at time $t = 0$ the velocity is $v = 0$ and the displacement is $x = 1$.



$$F = -12 \cos 2t$$

First we find the acceleration of the object in terms of the time t .

The mass of the object is 3 kg, so $m = 3$.

Hence, by Newton's second law, the acceleration a (in m s^{-2}) is given by

$$a = \frac{F}{m} = \frac{-12 \cos 2t}{3} = -4 \cos 2t.$$

Now we find the velocity v (in m s^{-1}) of the object in terms of the time t . The velocity is given by

$$\begin{aligned} v &= \int a \, dt = - \int 4 \cos 2t \, dt \\ &= -2 \sin 2t + c, \end{aligned}$$

where c is a constant.

The object starts from rest, so $v = 0$ when $t = 0$. Substituting these values into the equation above gives

$$0 = c,$$

so

$$v = -2 \sin 2t.$$

Finally, we find the position x (in metres) of the object. The position is given by

$$\begin{aligned} x &= \int v \, dt = - \int 2 \sin 2t \, dt \\ &= \cos 2t + d, \end{aligned}$$

where d is a constant.

Now, $x = 1$ when $t = 0$, and substituting these values into the equation above gives

$$1 = 1 + d.$$

Hence $d = 0$, so the position x (in metres) of the object at time t is given by

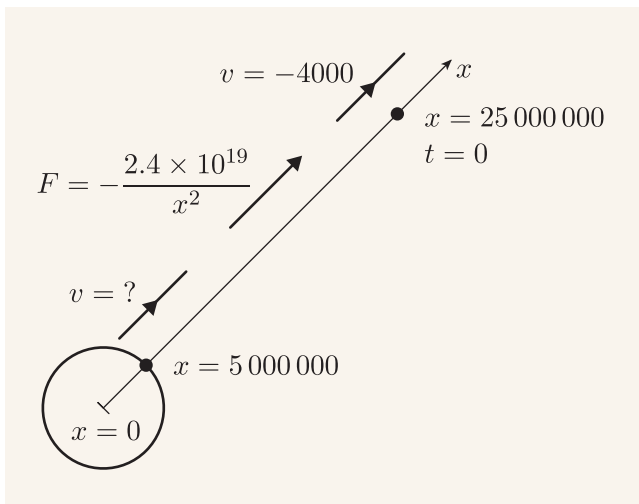
$$x = \cos 2t.$$

Solution to Exercise 14

Model the asteroid as a particle.

The x -axis points in the direction from the planet to the asteroid, with the origin at the centre of the planet, as indicated in the question.

At time $t = 0$, the velocity is $v = -4000$ and the position is $x = 25\,000\,000$.



First we find the acceleration of the object in terms of the time t .

The mass of the object is $20\,000$ kg, so $m = 20\,000$. Hence, by Newton's second law, the acceleration a (in ms^{-2}) is given by

$$a = \frac{F}{m} = \frac{\left(-\frac{2.4 \times 10^{19}}{x^2}\right)}{20\,000} = -\frac{1.2 \times 10^{15}}{x^2}.$$

This expresses the acceleration as a function of the position x .

To find the velocity v (in ms^{-1}) of the object as a function of x , we use the equation

$$a = v \frac{dv}{dx},$$

which gives

$$v \frac{dv}{dx} = -\frac{1.2 \times 10^{15}}{x^2}.$$

This is a separable differential equation.

Separating the variables gives

$$\int v \, dv = - \int \frac{1.2 \times 10^{15}}{x^2} \, dx,$$

so

$$\frac{v^2}{2} = \frac{1.2 \times 10^{15}}{x} + c,$$

where c is a constant.

Using the initial condition that $v = -4000$ when $x = 25\,000\,000$ gives

$$\frac{(-4000)^2}{2} = \frac{1.2 \times 10^{15}}{25\,000\,000} + c,$$

so

$$8\,000\,000 = \frac{1200 \times 10^{12}}{25 \times 10^6} + c,$$

thus

$$8\,000\,000 = 48 \times 10^6 + c,$$

hence

$$8\,000\,000 = 48\,000\,000 + c,$$

so

$$c = -40\,000\,000,$$

that is, $c = -4 \times 10^7$. Hence

$$\frac{v^2}{2} = \frac{1.2 \times 10^{15}}{x} - 4 \times 10^7,$$

that is,

$$v^2 = \frac{2.4 \times 10^{15}}{x} - 8 \times 10^7.$$

Since the asteroid moves towards the planet (in the negative direction), its velocity v is always negative, so

$$v = -\sqrt{\frac{2.4 \times 10^{15}}{x} - 8 \times 10^7}.$$

The asteroid hits the planet when $x = 5\,000\,000 = 5 \times 10^6$, which gives

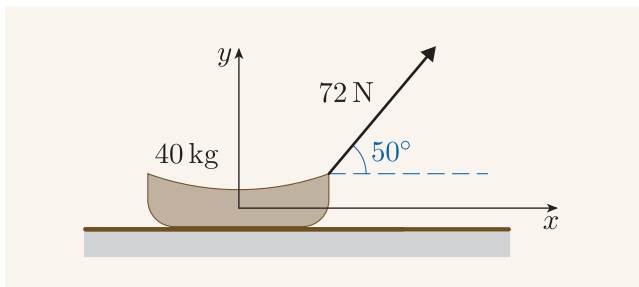
$$\begin{aligned} v &= -\sqrt{\frac{2.4 \times 10^{15}}{5 \times 10^6} - 8 \times 10^7} \\ &= -\sqrt{\frac{240 \times 10^{13}}{5 \times 10^6} - 8 \times 10^7} \\ &= -\sqrt{48 \times 10^7 - 8 \times 10^7} \\ &= -\sqrt{40 \times 10^7} \\ &= -\sqrt{400\,000\,000} \\ &= -20\,000. \end{aligned}$$

Hence the speed of the asteroid on impact with the planet is $20\,000 \text{ ms}^{-1}$, that is, 20 km s^{-1} .

Solution to Exercise 15

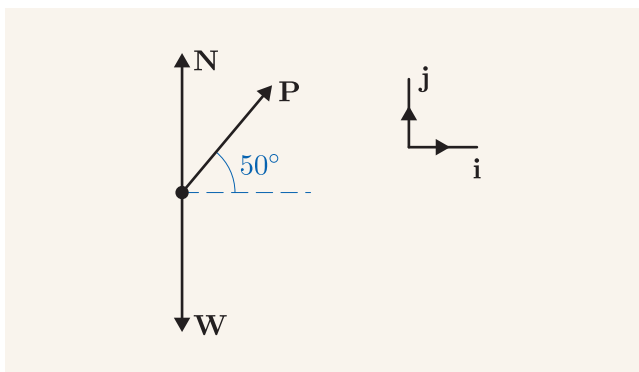
Model the sled as a particle.

A diagram of the situation is shown below.



Take the x -axis to be horizontal, pointing in the direction of the motion, and take the y -axis to be vertical, with the origin at the initial location of the sled.

The forces acting on the sled are its weight $\mathbf{W}$, the normal reaction $\mathbf{N}$ from the surface, and the pulling force, say $\mathbf{P}$. A force diagram is shown below.



We know that $|\mathbf{P}| = 72$ and $|\mathbf{W}| = 40g$, where g is the magnitude of the acceleration due to gravity. Let $N = |\mathbf{N}|$. Expressing the forces in component form gives

$$\mathbf{N} = N\mathbf{j},$$

$$\mathbf{P} = 72 \cos 50^\circ \mathbf{i} + 72 \sin 50^\circ \mathbf{j},$$

$$\mathbf{W} = -40g\mathbf{j}.$$

The resultant force $\mathbf{F}$ acting on the sled is given by

$$\begin{aligned}\mathbf{F} &= \mathbf{N} + \mathbf{P} + \mathbf{W} \\ &= N\mathbf{j} + 72 \cos 50^\circ \mathbf{i} + 72 \sin 50^\circ \mathbf{j} - 40g\mathbf{j} \\ &= 72 \cos 50^\circ \mathbf{i} + (N + 72 \sin 50^\circ - 40g)\mathbf{j}.\end{aligned}$$

Since the sled moves along the x -axis, its acceleration has component form

$$\mathbf{a} = a\mathbf{i},$$

where a is the $\mathbf{i}$ -component of the acceleration.

Newton's second law, $\mathbf{F} = m\mathbf{a}$, gives

$$40a\mathbf{i} = 72 \cos 50^\circ \mathbf{i} + (N + 72 \sin 50^\circ - 40g)\mathbf{j}.$$

Resolving this equation in the $\mathbf{i}$ -direction gives

$$40a = 72 \cos 50^\circ,$$

so

$$a = \frac{72 \cos 50^\circ}{40} = 1.1570 \dots$$

The magnitude of the acceleration of the sled is 1.2 m s^{-2} (to 2 s.f.).

Since the acceleration is constant, we can use an equation of motion for constant acceleration to find the speed after 5 seconds. We know that $v_0 = 0$, $a = 1.1570 \dots$ and $t = 5$, and we want to find v . Using $v = v_0 + at$ gives

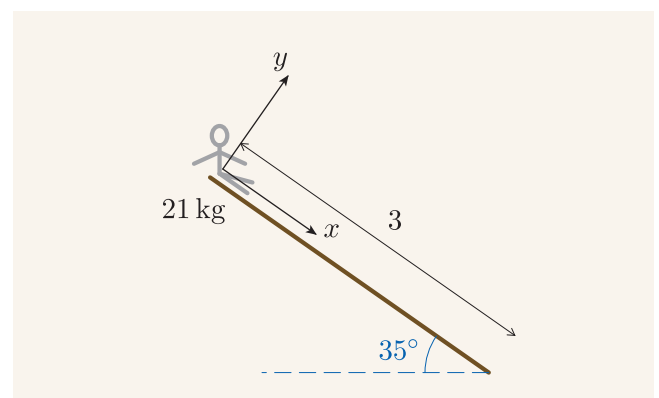
$$v = 0 + 1.1570 \dots \times 5 = 5.7850 \dots$$

The speed of the sled after 5 seconds is 5.8 m s^{-1} (to 2 s.f.).

Solution to Exercise 16

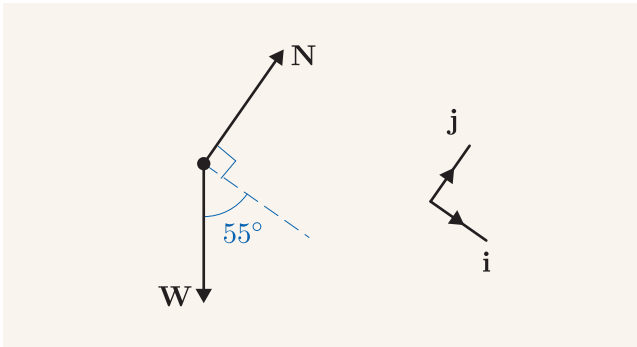
Model the child as a particle.

A diagram of the situation is shown below.



Take the x -axis to point down the slide and the y -axis to be perpendicular to the slide, with the origin at the top of the slide.

The forces acting on the child are her weight $\mathbf{W}$ and the normal reaction $\mathbf{N}$ from the slide. A force diagram is shown below.



We know that $|\mathbf{W}| = 21g$, where g is the magnitude of the acceleration due to gravity. Let $N = |\mathbf{N}|$. Expressing the forces in component form gives

$$\mathbf{N} = N\mathbf{j}$$

$$\mathbf{W} = 21g \cos 55^\circ \mathbf{i} - 21g \sin 55^\circ \mathbf{j}.$$

The resultant force $\mathbf{F}$ acting on the child is given by

$$\begin{aligned}\mathbf{F} &= \mathbf{N} + \mathbf{W} \\ &= N\mathbf{j} + 21g \cos 55^\circ \mathbf{i} - 21g \sin 55^\circ \mathbf{j} \\ &= 21g \cos 55^\circ \mathbf{i} + (N - 21g \sin 55^\circ) \mathbf{j}.\end{aligned}$$

Since the child moves along the x -axis, her acceleration $\mathbf{a}$ has component form

$$\mathbf{a} = a\mathbf{i},$$

where a is the $\mathbf{i}$ -component of the acceleration.

Newton's second law, $\mathbf{F} = m\mathbf{a}$, gives

$$21a\mathbf{i} = 21g \cos 55^\circ \mathbf{i} + (N - 21g \sin 55^\circ) \mathbf{j}.$$

Resolving this equation in the $\mathbf{i}$ -direction gives

$$21a = 21g \cos 55^\circ,$$

so

$$a = g \cos 55^\circ = 5.6210 \dots$$

The magnitude of the child's acceleration is 5.6 m s^{-2} (to 2 s.f.).

Since the acceleration is constant, we can use an equation of motion for constant acceleration to find the time taken to travel 3 metres. We know that $v_0 = 0$, $a = 5.6210 \dots$ and $x = 3$, and we want to find t .

Using $x = v_0 t + \frac{1}{2} a t^2$ gives

$$3 = 0t + \frac{1}{2} \times 5.6210 \dots \times t^2,$$

which we can rearrange to get

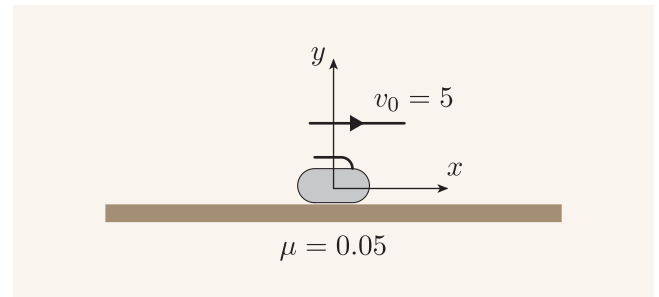
$$t = \sqrt{\frac{2 \times 3}{5.6210 \dots}} = 1.033 \dots$$

The child takes 1.0 seconds (to 2 s.f.) to reach the bottom of the slide.

Solution to Exercise 17

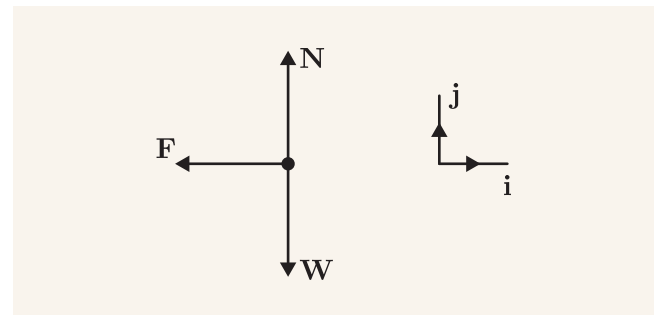
Model the stone as a particle.

A diagram of the situation is shown below.



Take the x -axis to be horizontal, pointing in the direction of motion, and take the y -axis to be vertical, with the origin at the initial location of the stone.

The forces acting on the stone are its weight $\mathbf{W}$, the normal reaction $\mathbf{N}$ of the surface on the stone, and the friction force $\mathbf{F}$, which acts in the negative x -direction. A force diagram is shown below.



Let m be the mass of the stone, and let $N = |\mathbf{N}|$. Then $|\mathbf{W}| = mg$, where g is the magnitude of the acceleration due to gravity. Also, we have $|\mathbf{F}| = \mu|\mathbf{N}|$, so $|\mathbf{F}| = \mu N$, where $\mu = 0.05$ is the coefficient of sliding friction.

Expressing the forces in component form gives

$$\mathbf{N} = N\mathbf{j},$$

$$\mathbf{F} = -\mu N\mathbf{i},$$

$$\mathbf{W} = -mg\mathbf{j}.$$

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The resultant force $\mathbf{R}$ acting on the stone is given by

$$\begin{aligned}\mathbf{R} &= \mathbf{N} + \mathbf{F} + \mathbf{W} \\ &= N\mathbf{j} - \mu N\mathbf{i} - mg\mathbf{j} \\ &= -\mu N\mathbf{i} + (N - mg)\mathbf{j}.\end{aligned}$$

Since the stone moves along the x -axis, its acceleration has component form

$$\mathbf{a} = a\mathbf{i},$$

where a is the $\mathbf{i}$ -component of the acceleration.

Newton's second law gives

$$ma\mathbf{i} = -\mu N\mathbf{i} + (N - mg)\mathbf{j}.$$

Resolving in the $\mathbf{i}$ - and $\mathbf{j}$ -directions gives

$$\begin{aligned}ma &= -\mu N, \\ 0 &= N - mg.\end{aligned}$$

The second equation gives

$$N = mg.$$

Substituting into the first equation gives

$$ma = -\mu mg,$$

so

$$a = -\mu g = -0.05 \times 9.8 = -0.49.$$

So we know that the stone has initial velocity $v_0 = 5$, and constant acceleration $a = -0.49$. It will stop when it has final velocity $v = 0$.

Using $v^2 = v_0^2 + 2ax$ gives

$$0^2 = 5^2 + 2(-0.49)x,$$

which we can rearrange to give

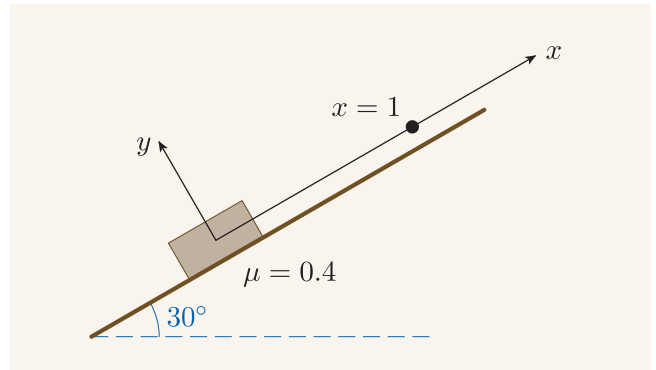
$$x = \frac{5^2}{2 \times 0.49} = 25.51 \dots$$

The stone will therefore travel 26 metres (to 2 s.f.) before coming to rest. Since the stone aimed at is 30 metres away, it will not hit this stone.

Solution to Exercise 18

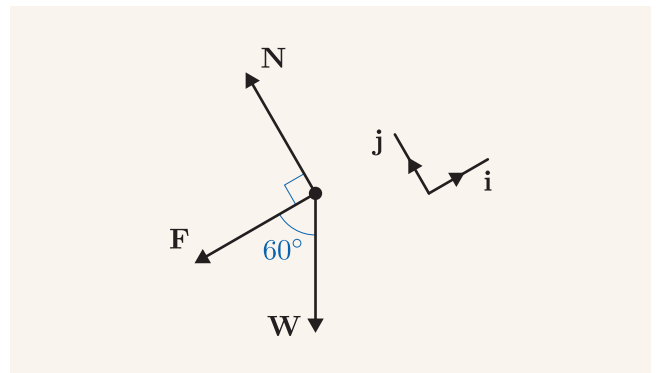
Model the box as a particle.

A diagram of the situation is shown below.



Take the x -axis to point up the slope and the y -axis to be perpendicular to the slope, with the origin at the point at which the box is given the initial push.

The forces acting on the box are its weight $\mathbf{W}$, the normal reaction $\mathbf{N}$ from the slope, and the friction force $\mathbf{F}$. The friction force $\mathbf{F}$ acts against the motion, so it acts down the slope in the negative x -direction. A force diagram is shown below.



Let m be the mass of the box, and let $N = |\mathbf{N}|$. Then $|\mathbf{W}| = mg$, where g is the magnitude of the acceleration due to gravity. Also, since $|\mathbf{F}| = \mu|\mathbf{N}|$, we have $|\mathbf{F}| = \mu N$, where $\mu = 0.4$ is the coefficient of sliding friction. Expressing the forces in component form gives

$$\begin{aligned}\mathbf{N} &= N\mathbf{j}, \\ \mathbf{W} &= -mg \cos 60^\circ \mathbf{i} - mg \sin 60^\circ \mathbf{j} \\ &= -\frac{1}{2}mg\mathbf{i} - \frac{\sqrt{3}}{2}mg\mathbf{j}, \\ \mathbf{F} &= -\mu N\mathbf{i}.\end{aligned}$$

The resultant force $\mathbf{R}$ acting on the box is given by

$$\begin{aligned}\mathbf{R} &= \mathbf{N} + \mathbf{W} + \mathbf{F} \\ &= N\mathbf{j} - \frac{1}{2}mg\mathbf{i} - \frac{\sqrt{3}}{2}mg\mathbf{j} - \mu N\mathbf{i} \\ &= -\left(\frac{1}{2}mg + \mu N\right)\mathbf{i} + \left(N - \frac{\sqrt{3}}{2}mg\right)\mathbf{j}.\end{aligned}$$

Since the box moves along the x -axis, its acceleration $\mathbf{a}$ has component form

$$\mathbf{a} = a \mathbf{i},$$

where a is the $\mathbf{i}$ -component of the acceleration.

Newton's second law gives

$$ma \mathbf{i} = - \left(\frac{1}{2}mg + \mu N \right) \mathbf{i} + \left(N - \frac{\sqrt{3}}{2}mg \right) \mathbf{j}.$$

Resolving this equation in the $\mathbf{i}$ - and $\mathbf{j}$ -directions gives

$$ma = - \left(\frac{1}{2}mg + \mu N \right),$$

$$0 = N - \frac{\sqrt{3}}{2}mg.$$

Rearranging the second equation gives

$$N = \frac{\sqrt{3}}{2}mg,$$

which we can substitute into the first equation to obtain

$$ma = - \left(\frac{1}{2}mg + \mu \frac{\sqrt{3}}{2}mg \right),$$

so

$$\begin{aligned} a &= -\frac{g}{2} (1 + \sqrt{3}\mu) \\ &= -\frac{9.8}{2} (1 + \sqrt{3} \times 0.4) \\ &= -8.2948 \dots \end{aligned}$$

So we know that the constant acceleration is $a = -8.2948 \dots$ and the final velocity is $v = 0$, and we want to find the value of v_0 that gives $x = 1$.

Using $v^2 = v_0^2 + 2ax$ gives

$$0^2 = v_0^2 + 2 \times (-8.2948 \dots) \times 1,$$

which we can rearrange to give

$$v_0^2 = 2 \times 8.2948 \dots,$$

so

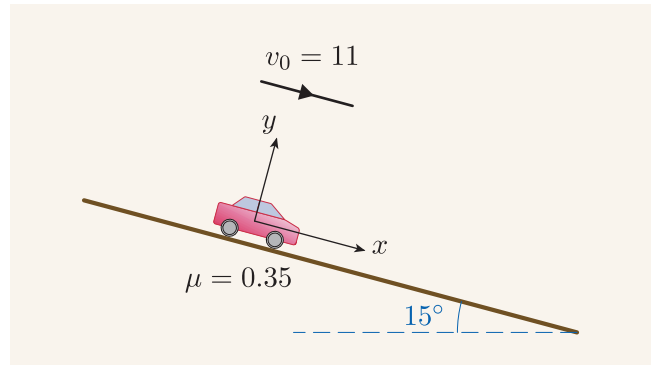
$$v_0 = 4.073 \dots$$

Hence the initial speed needed to ensure that the box travels one metre up the slope is 4.1 m s^{-1} (to 2 s.f.).

Solution to Exercise 19

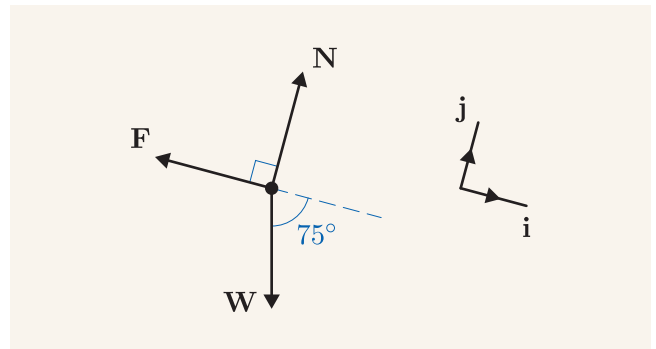
Model the car as a particle.

A diagram of the situation is shown below.



Take the x -axis to point down the ramp and the y -axis to be perpendicular to the ramp, with the origin at the point at which the brakes of the car lock.

The forces acting on the car are its weight $\mathbf{W}$, the normal reaction $\mathbf{N}$ from the slope, and the friction force $\mathbf{F}$. The friction force $\mathbf{F}$ acts against the motion, so it acts up the ramp in the negative x -direction. A force diagram is shown below.



Let m be the mass of the car, and let $N = |\mathbf{N}|$. Then $|\mathbf{W}| = mg$, where g is the magnitude of the acceleration due to gravity. Also, since $|\mathbf{F}| = \mu|\mathbf{N}|$, we have $|\mathbf{F}| = \mu N$, where $\mu = 0.35$ is the coefficient of sliding friction. Expressing the forces in component form gives

$$\mathbf{N} = N \mathbf{j},$$

$$\mathbf{W} = mg \cos 75^\circ \mathbf{i} - mg \sin 75^\circ \mathbf{j},$$

$$\mathbf{F} = -\mu N \mathbf{i}.$$

The resultant force $\mathbf{R}$ acting on the car is given by

$$\begin{aligned} \mathbf{R} &= \mathbf{N} + \mathbf{W} + \mathbf{F} \\ &= N \mathbf{j} + mg \cos 75^\circ \mathbf{i} \\ &\quad - mg \sin 75^\circ \mathbf{j} - \mu N \mathbf{i} \\ &= (mg \cos 75^\circ - \mu N) \mathbf{i} \\ &\quad + (N - mg \sin 75^\circ) \mathbf{j}. \end{aligned}$$

Exercise Booklet 10

Since the car moves along the x -axis, its acceleration $\mathbf{a}$ has component form

$$\mathbf{a} = a \mathbf{i},$$

where a is the $\mathbf{i}$ -component of the acceleration.

Newton's second law gives

$$ma \mathbf{i} = (mg \cos 75^\circ - \mu N) \mathbf{i} + (N - mg \sin 75^\circ) \mathbf{j}.$$

Resolving this equation in the $\mathbf{i}$ - and $\mathbf{j}$ -directions gives

$$ma = mg \cos 75^\circ - \mu N, \\ 0 = N - mg \sin 75^\circ.$$

Rearranging the second equation gives

$$N = mg \sin 75^\circ,$$

which we can substitute into the first equation to obtain

$$ma = mg \cos 75^\circ - \mu mg \sin 75^\circ,$$

which gives

$$a = g(\cos 75^\circ - \mu \sin 75^\circ) \\ = 9.8(\cos 75^\circ - 0.35 \times \sin 75^\circ) \\ = -0.7766 \dots$$

So we know that the constant acceleration is $a = -0.7766 \dots$ and the initial velocity is $v_0 = 11$, and we want to find v when $t = 5$.

Using $v = v_0 + at$ gives

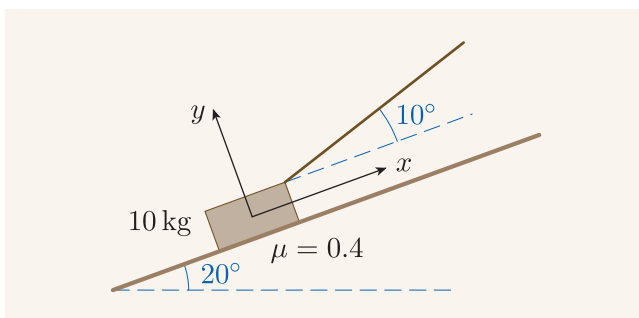
$$v = 11 - 0.7766 \dots \times 5 = 7.1165 \dots$$

Hence the speed of the car when the driver regains control is 7.1 m s^{-1} (to 2 s.f.).

Solution to Exercise 20

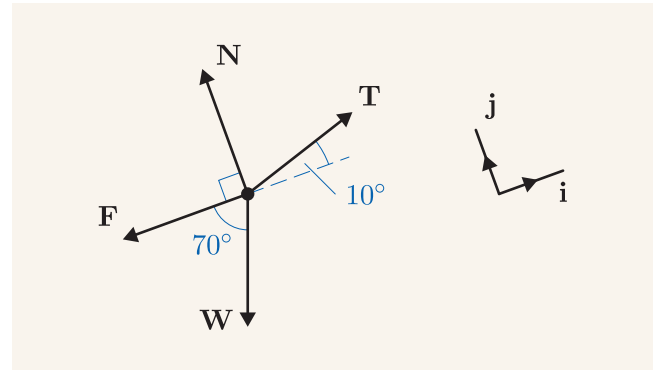
Model the box as a particle.

A diagram of the situation is shown below.



Take the x -axis to point up the slope and the y -axis to be perpendicular to the slope, with the origin at the point at which the motion begins.

The forces acting on the box are its weight $\mathbf{W}$, the normal reaction $\mathbf{N}$ from the slope, the pulling force $\mathbf{T}$ from the rope, and the friction force $\mathbf{F}$. The friction force $\mathbf{F}$ acts against the motion, so it acts down the slope in the negative x -direction. A force diagram is shown below.



We know that the mass of the box is $m = 10$, so $|\mathbf{W}| = 10g$, where g is the magnitude of the acceleration due to gravity. Let $N = |\mathbf{N}|$. Since $|\mathbf{F}| = \mu|\mathbf{N}|$, we have $|\mathbf{F}| = \mu N$, where $\mu = 0.4$ is the coefficient of sliding friction. We also have $|\mathbf{T}| = 80$. Expressing the forces in component form gives

$$\mathbf{N} = N \mathbf{j}, \\ \mathbf{W} = -10g \cos 70^\circ \mathbf{i} - 10g \sin 70^\circ \mathbf{j}, \\ \mathbf{T} = 80 \cos 10^\circ \mathbf{i} + 80 \sin 10^\circ \mathbf{j}, \\ \mathbf{F} = -\mu N \mathbf{i}.$$

The resultant force $\mathbf{R}$ acting on the box is given by

$$\mathbf{R} = \mathbf{N} + \mathbf{W} + \mathbf{T} + \mathbf{F} \\ = N \mathbf{j} - 10g \cos 70^\circ \mathbf{i} - 10g \sin 70^\circ \mathbf{j} \\ + 80 \cos 10^\circ \mathbf{i} + 80 \sin 10^\circ \mathbf{j} - \mu N \mathbf{i} \\ = (80 \cos 10^\circ - 10g \cos 70^\circ - \mu N) \mathbf{i} \\ + (80 \sin 10^\circ - 10g \sin 70^\circ + N) \mathbf{j}.$$

Since the box moves along the x -axis, its acceleration $\mathbf{a}$ has component form

$$\mathbf{a} = a \mathbf{i},$$

where a is the $\mathbf{i}$ -component of the acceleration.

Newton's second law gives

$$10a \mathbf{i} = (80 \cos 10^\circ - 10g \cos 70^\circ - \mu N) \mathbf{i} + (80 \sin 10^\circ - 10g \sin 70^\circ + N) \mathbf{j}.$$

Resolving this equation in the $\mathbf{i}$ - and $\mathbf{j}$ -directions gives

$$10a = 80 \cos 10^\circ - 10g \cos 70^\circ - \mu N, \\ 0 = 80 \sin 10^\circ - 10g \sin 70^\circ + N.$$

Rearranging the second equation gives

$$N = 10g \sin 70^\circ - 80 \sin 10^\circ,$$

which we can substitute into the first equation to obtain

$$10a = 80 \cos 10^\circ - 10g \cos 70^\circ - \mu(10g \sin 70^\circ - 80 \sin 10^\circ),$$

which gives

$$\begin{aligned} a &= 8 \cos 10^\circ - g \cos 70^\circ \\ &\quad - \mu(g \sin 70^\circ - 8 \sin 10^\circ) \\ &= 8 \cos 10^\circ - 9.8 \cos 70^\circ \\ &\quad - 0.4(9.8 \sin 70^\circ - 8 \sin 10^\circ) \\ &= 1.3987 \dots \end{aligned}$$

Hence the acceleration of the box up the slope is 1.4 m s^{-2} (to 2 s.f.).

Solution to Exercise 21

(a) When $t = 0$, the position is

$$\begin{aligned} \mathbf{r} &= (0.5 \times 0^3) \mathbf{i} + (2 \times 0 + 4) \mathbf{j} \\ &\quad - (0.25 \times 0^2) \mathbf{k} \\ &= 4 \mathbf{j}. \end{aligned}$$

The distance from the origin is

$$|\mathbf{r}| = \sqrt{0^2 + 4^2 + 0^2} = 4.$$

(b) When $t = 2$, the position is

$$\begin{aligned} \mathbf{r} &= (0.5 \times 2^3) \mathbf{i} + (2 \times 2 + 4) \mathbf{j} \\ &\quad - (0.25 \times 2^2) \mathbf{k} \\ &= 4 \mathbf{i} + 8 \mathbf{j} - \mathbf{k}. \end{aligned}$$

The distance from the origin is

$$|\mathbf{r}| = \sqrt{4^2 + 8^2 + (-1)^2} = \sqrt{81} = 9.$$

(c) When $t = 10$, the position is

$$\begin{aligned} \mathbf{r} &= (0.5 \times 10^3) \mathbf{i} + (2 \times 10 + 4) \mathbf{j} \\ &\quad - (0.25 \times 10^2) \mathbf{k} \\ &= 500 \mathbf{i} + 24 \mathbf{j} - 25 \mathbf{k}. \end{aligned}$$

The distance from the origin is

$$\begin{aligned} |\mathbf{r}| &= \sqrt{500^2 + 24^2 + (-25)^2} \\ &= \sqrt{251\,201} \\ &= 501.20 \quad (\text{to } 2 \text{ d.p.}). \end{aligned}$$

Solution to Exercise 22

(a) The velocity of the particle is given by

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(0.5t^3) \mathbf{i} + \frac{d}{dt}(2t + 4) \mathbf{j} \\ &\quad - \frac{d}{dt}(0.25t^2) \mathbf{k} \\ &= 1.5t^2 \mathbf{i} + 2 \mathbf{j} - 0.5t \mathbf{k}. \end{aligned}$$

(b) When $t = 10$, the velocity is

$$\begin{aligned} \mathbf{v} &= (1.5 \times 10^2) \mathbf{i} + 2 \mathbf{j} - (0.5 \times 10) \mathbf{k} \\ &= 150 \mathbf{i} + 2 \mathbf{j} - 5 \mathbf{k}. \end{aligned}$$

So the speed when $t = 10$ is

$$\begin{aligned} |\mathbf{v}| &= \sqrt{150^2 + 2^2 + (-5)^2} \\ &= \sqrt{22\,529} \\ &= 150.10 \quad (\text{to } 2 \text{ d.p.}). \end{aligned}$$

Solution to Exercise 23

The position of the particle at time t is given by

$$\begin{aligned} \mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int (2e^{-t} \mathbf{i} + 2t^3 \mathbf{j} - t \mathbf{k}) \, dt \\ &= \left(\int 2e^{-t} \, dt \right) \mathbf{i} + \left(\int 2t^3 \, dt \right) \mathbf{j} \\ &\quad - \left(\int t \, dt \right) \mathbf{k} \\ &= -2e^{-t} \mathbf{i} + \frac{1}{2}t^4 \mathbf{j} - \frac{1}{2}t^2 \mathbf{k} + \mathbf{c}, \end{aligned}$$

where $\mathbf{c}$ is a vector constant.

We know that $\mathbf{r} = \frac{1}{2} \mathbf{j}$ when $t = 0$.

Substituting these values into the equation above gives

$$\frac{1}{2} \mathbf{j} = -2e^{-0} \mathbf{i} + \left(\frac{1}{2} \times 0^4\right) \mathbf{j} - \left(\frac{1}{2} \times 0^2\right) \mathbf{k} + \mathbf{c},$$

so $\mathbf{c} = 2 \mathbf{i} + \frac{1}{2} \mathbf{j}$. Thus

$$\begin{aligned} \mathbf{r} &= -2e^{-t} \mathbf{i} + \frac{1}{2}t^4 \mathbf{j} - \frac{1}{2}t^2 \mathbf{k} + 2 \mathbf{i} + \frac{1}{2} \mathbf{j} \\ &= 2(1 - e^{-t}) \mathbf{i} + \frac{1}{2}(1 + t^4) \mathbf{j} - \frac{1}{2}t^2 \mathbf{k}. \end{aligned}$$

The position of the particle at time $t = 4$ is

$$\begin{aligned} \mathbf{r} &= 2(1 - e^{-4}) \mathbf{i} + \frac{1}{2}(1 + 4^4) \mathbf{j} - \left(\frac{1}{2} \times 4^2\right) \mathbf{k} \\ &= 2(1 - e^{-4}) \mathbf{i} + \frac{257}{2} \mathbf{j} - 8 \mathbf{k}. \end{aligned}$$

The distance from the origin at time $t = 4$ is therefore

$$\begin{aligned} |\mathbf{r}| &= \sqrt{(2(1 - e^{-4}))^2 + \left(\frac{257}{2}\right)^2 + (-8)^2} \\ &= \sqrt{16\,580.104 \dots} \\ &= 128.76 \quad (\text{to } 2 \text{ d.p.}). \end{aligned}$$

Solution to Exercise 24

Model the child as a particle.

- (a) The velocity of the child at time t is given by

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt} \left(2 \cos \frac{2\pi t}{5} \right) \mathbf{i} + \frac{d}{dt} \left(2 \sin \frac{2\pi t}{5} \right) \mathbf{j} \\ &\quad + \frac{d}{dt} \left(12 - \frac{4t}{5} \right) \mathbf{k} \\ &= -\frac{4\pi}{5} \sin \frac{2\pi t}{5} \mathbf{i} + \frac{4\pi}{5} \cos \frac{2\pi t}{5} \mathbf{j} - \frac{4}{5} \mathbf{k}.\end{aligned}$$

- (b) The velocity of the child when $t = 5$ is

$$\begin{aligned}\mathbf{v} &= -\frac{4\pi}{5} \sin \frac{2\pi \times 5}{5} \mathbf{i} \\ &\quad + \frac{4\pi}{5} \cos \frac{2\pi \times 5}{5} \mathbf{j} - \frac{4}{5} \mathbf{k} \\ &= \frac{4\pi}{5} \mathbf{j} - \frac{4}{5} \mathbf{k},\end{aligned}$$

so the speed when $t = 5$ is

$$\begin{aligned}|\mathbf{v}| &= \sqrt{\left(\frac{4\pi}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} \\ &= \sqrt{\frac{16(\pi^2 + 1)}{25}} \\ &= \frac{4}{5} \sqrt{\pi^2 + 1} = 2.6375 \dots\end{aligned}$$

The speed of the child after 5 seconds is 2.64 m s^{-1} (to 3 s.f.).

- (c) The acceleration of the child at time t is given by

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt} \left(-\frac{4\pi}{5} \sin \frac{2\pi t}{5} \right) \mathbf{i} \\ &\quad + \frac{d}{dt} \left(\frac{4\pi}{5} \cos \frac{2\pi t}{5} \right) \mathbf{j} + \frac{d}{dt} \left(-\frac{4}{5} \right) \mathbf{k} \\ &= -\frac{8\pi^2}{25} \cos \frac{2\pi t}{5} \mathbf{i} - \frac{8\pi^2}{25} \sin \frac{2\pi t}{5} \mathbf{j}.\end{aligned}$$

Solution to Exercise 25

Model the skier as a particle.

- (a) The velocity $\mathbf{v}$ of the skier is given by

$$\begin{aligned}\mathbf{v} &= \int \mathbf{a} dt \\ &= \int \left(\frac{1}{2} \mathbf{i} - \pi^2 \sin \pi t \mathbf{j} \right) dt \\ &= \left(\int \frac{1}{2} dt \right) \mathbf{i} - \left(\int \pi^2 \sin \pi t dt \right) \mathbf{j} \\ &= \frac{1}{2} t \mathbf{i} + \pi \cos \pi t \mathbf{j} + \mathbf{c},\end{aligned}$$

where $\mathbf{c}$ is a vector constant.

We know that $\mathbf{v} = \pi \mathbf{j} - \frac{1}{2} \mathbf{k}$ when $t = 0$. Substituting these values into the equation above gives

$$\pi \mathbf{j} - \frac{1}{2} \mathbf{k} = \left(\frac{1}{2} \times 0 \right) \mathbf{i} + \pi \cos(\pi \times 0) \mathbf{j} + \mathbf{c},$$

that is,

$$\pi \mathbf{j} - \frac{1}{2} \mathbf{k} = \pi \mathbf{j} + \mathbf{c}.$$

So $\mathbf{c} = -\frac{1}{2} \mathbf{k}$. This gives

$$\mathbf{v} = \frac{1}{2} t \mathbf{i} + \pi \cos \pi t \mathbf{j} - \frac{1}{2} \mathbf{k}.$$

- (b) The position $\mathbf{r}$ of the skier is given by

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v} dt \\ &= \int \left(\frac{1}{2} t \mathbf{i} + \pi \cos \pi t \mathbf{j} - \frac{1}{2} \mathbf{k} \right) dt \\ &= \left(\int \frac{1}{2} t dt \right) \mathbf{i} + \left(\int \pi \cos \pi t dt \right) \mathbf{j} \\ &\quad - \left(\int \frac{1}{2} dt \right) \mathbf{k} \\ &= \frac{1}{4} t^2 \mathbf{i} + \sin \pi t \mathbf{j} - \frac{1}{2} t \mathbf{k} + \mathbf{d},\end{aligned}$$

where $\mathbf{d}$ is a vector constant.

We know that $\mathbf{r} = 4 \mathbf{k}$ when $t = 0$. Substituting these values into the equation above gives

$$\begin{aligned}4 \mathbf{k} &= \left(\frac{1}{4} \times 0^2 \right) \mathbf{i} + \sin(\pi \times 0) \mathbf{j} \\ &\quad - \left(\frac{1}{2} \times 0 \right) \mathbf{k} + \mathbf{d},\end{aligned}$$

that is,

$$4 \mathbf{k} = \mathbf{d}.$$

So $\mathbf{d} = 4 \mathbf{k}$. This gives

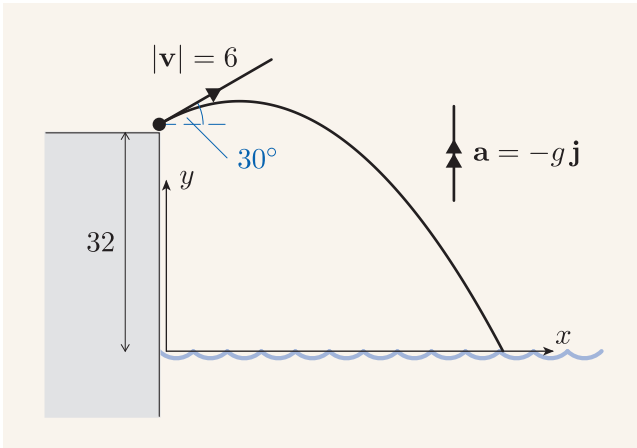
$$\begin{aligned}\mathbf{r} &= \frac{1}{4} t^2 \mathbf{i} + \sin \pi t \mathbf{j} - \frac{1}{2} t \mathbf{k} + 4 \mathbf{k} \\ &= \frac{1}{4} t^2 \mathbf{i} + \sin \pi t \mathbf{j} + \left(4 - \frac{1}{2} t \right) \mathbf{k}.\end{aligned}$$

Solution to Exercise 26

Model the pebble as a particle.

Measure time from the moment when the pebble is released.

Choose the x -axis to be horizontal, pointing out to sea, and choose the y -axis to point vertically upwards, with the origin at the foot of the cliff directly below where the pebble is released.



We know that at time $t = 0$, the position is $\mathbf{r} = 32\mathbf{j}$, and the velocity is

$$\begin{aligned}\mathbf{v} &= 6 \cos 30^\circ \mathbf{i} + 6 \sin 30^\circ \mathbf{j} \\ &= 6 \times \frac{\sqrt{3}}{2} \mathbf{i} + 6 \times \frac{1}{2} \mathbf{j} \\ &= 3\sqrt{3}\mathbf{i} + 3\mathbf{j}.\end{aligned}$$

The acceleration $\mathbf{a}$ is the acceleration due to gravity, so it has magnitude g and points downwards. This gives $\mathbf{a} = -g\mathbf{j}$.

The velocity of the pebble is given by

$$\mathbf{v} = \int \mathbf{a} \, dt = - \int g\mathbf{j} \, dt = -gt\mathbf{j} + \mathbf{c},$$

where $\mathbf{c}$ is a vector constant.

At time $t = 0$, the velocity is $3\sqrt{3}\mathbf{i} + 3\mathbf{j}$. Substituting these values into the equation above gives

$$3\sqrt{3}\mathbf{i} + 3\mathbf{j} = -(g \times 0)\mathbf{j} + \mathbf{c},$$

so $\mathbf{c} = 3\sqrt{3}\mathbf{i} + 3\mathbf{j}$.

This gives

$$\begin{aligned}\mathbf{v} &= -gt\mathbf{j} + 3\sqrt{3}\mathbf{i} + 3\mathbf{j} \\ &= 3\sqrt{3}\mathbf{i} + (3 - gt)\mathbf{j}.\end{aligned}$$

The position of the pebble is given by

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int (3\sqrt{3}\mathbf{i} + (3 - gt)\mathbf{j}) \, dt \\ &= 3\sqrt{3}t\mathbf{i} + (3t - \tfrac{1}{2}gt^2)\mathbf{j} + \mathbf{d},\end{aligned}$$

where $\mathbf{d}$ is a vector constant.

At time $t = 0$, the position is $\mathbf{r} = 32\mathbf{j}$.

Substituting these values into the equation above gives

$$32\mathbf{j} = (3\sqrt{3} \times 0)\mathbf{i} + (3 \times 0 - \tfrac{1}{2}g \times 0^2)\mathbf{j} + \mathbf{d},$$

so $\mathbf{d} = 32\mathbf{j}$.

Hence the position of the pebble at time t is given by

$$\begin{aligned}\mathbf{r} &= 3\sqrt{3}t\mathbf{i} + (3t - \tfrac{1}{2}gt^2)\mathbf{j} + 32\mathbf{j} \\ &= 3\sqrt{3}t\mathbf{i} + (32 + 3t - \tfrac{1}{2}gt^2)\mathbf{j}.\end{aligned}$$

- (a) The pebble is at its maximum height when the $\mathbf{j}$ -component of the velocity is momentarily zero. This is when

$$3 - gt = 0,$$

which gives

$$t = \frac{3}{g}.$$

At this time, the $\mathbf{j}$ -component of the position is

$$\begin{aligned}32 + 3t - \tfrac{1}{2}gt^2 &= 32 + 3 \times \frac{3}{g} - \tfrac{1}{2}g \left(\frac{3}{g}\right)^2 \\ &= 32 + \frac{9}{g} - \frac{9}{2g} \\ &= 32 + \frac{9}{2g} \\ &= 32 + \frac{9}{2 \times 9.8} \\ &= 32.4591 \dots\end{aligned}$$

Hence the maximum height above the sea reached by the pebble is 32 metres (to 2 s.f.).

- (b) The pebble hits the sea when the $\mathbf{j}$ -component of the position is zero. This is when

$$32 + 3t - \tfrac{1}{2}gt^2 = 0.$$

This is a quadratic equation in t .

Multiplying it through by -1 gives

$$\tfrac{1}{2}gt^2 - 3t - 32 = 0.$$

Exercise Booklet 10

Applying the quadratic formula gives

$$t = \frac{3 \pm \sqrt{(-3)^2 - 4 \times \frac{1}{2}g(-32)}}{2 \times \frac{1}{2}g}$$

$$= \frac{3 \pm \sqrt{9 + 64 \times 9.8}}{9.8}.$$

Since taking the negative square root will result in a negative time, we choose the positive square root, to get

$$t = \frac{3 + \sqrt{9 + 64 \times 9.8}}{9.8} = 2.879 \dots$$

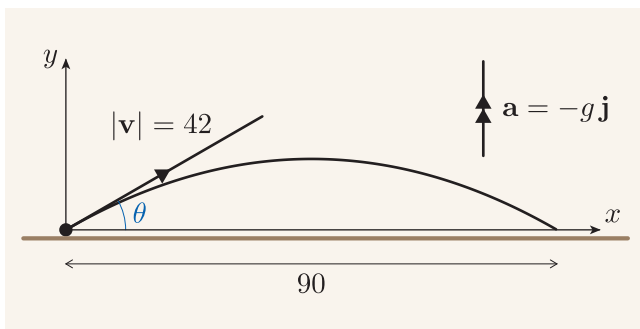
Hence the pebble hits the sea after 2.9 seconds (to 2 s.f.).

Solution to Exercise 27

Model the arrow as a particle.

Measure time from the moment when the arrow is released.

Choose the x -axis to be horizontal, pointing from the archer towards the centre of the target, and choose the y -axis to point vertically upwards, with the origin at the point from where the archer releases the arrow.



Let θ be the angle above the horizontal at which the archer shoots the arrow. It is between 0° and 90° .

We know that at time $t = 0$, the position is $\mathbf{r} = \mathbf{0}$ and the velocity is

$$\mathbf{v} = 42 \cos \theta \mathbf{i} + 42 \sin \theta \mathbf{j}.$$

The acceleration $\mathbf{a}$ is the acceleration due to gravity, so it has magnitude g and points downwards. This gives $\mathbf{a} = -g \mathbf{j}$.

The velocity of the arrow is given by

$$\mathbf{v} = \int \mathbf{a} \, dt = - \int g \mathbf{j} \, dt = -gt \mathbf{j} + \mathbf{c},$$

where $\mathbf{c}$ is a vector constant.

At time $t = 0$, the velocity is

$$\mathbf{v} = 42 \cos \theta \mathbf{i} + 42 \sin \theta \mathbf{j}.$$

Substituting these values into the equation above gives

$$42 \cos \theta \mathbf{i} + 42 \sin \theta \mathbf{j} = -(g \times 0) \mathbf{j} + \mathbf{c},$$

so

$$\mathbf{c} = 42 \cos \theta \mathbf{i} + 42 \sin \theta \mathbf{j}.$$

This gives

$$\mathbf{v} = -gt \mathbf{j} + 42 \cos \theta \mathbf{i} + 42 \sin \theta \mathbf{j}$$

$$= 42 \cos \theta \mathbf{i} + (42 \sin \theta - gt) \mathbf{j}.$$

The position of the arrow is given by

$$\mathbf{r} = \int \mathbf{v} \, dt$$

$$= \int (42 \cos \theta \mathbf{i} + (42 \sin \theta - gt) \mathbf{j}) \, dt$$

$$= 42t \cos \theta \mathbf{i} + (42t \sin \theta - \frac{1}{2}gt^2) \mathbf{j} + \mathbf{d},$$

where $\mathbf{d}$ is a vector constant.

At time $t = 0$, the position is $\mathbf{r} = \mathbf{0}$.

Substituting these values into the equation above gives

$$\mathbf{0} = (42 \times 0 \cos \theta) \mathbf{i}$$

$$+ (42 \times 0 \sin \theta - \frac{1}{2}g \times 0^2) \mathbf{j} + \mathbf{d},$$

so $\mathbf{d} = \mathbf{0}$.

Hence the position of the arrow at time t is given by

$$\mathbf{r} = 42t \cos \theta \mathbf{i} + (42t \sin \theta - \frac{1}{2}gt^2) \mathbf{j}.$$

The arrow will hit the centre of the target if its position is given by $\mathbf{r} = 90 \mathbf{i}$ at some time t . That is, we want

$$42t \cos \theta \mathbf{i} + (42t \sin \theta - \frac{1}{2}gt^2) \mathbf{j} = 90 \mathbf{i}.$$

Resolving this vector equation in the $\mathbf{i}$ - and $\mathbf{j}$ -directions gives

$$42t \cos \theta = 90,$$

$$42t \sin \theta - \frac{1}{2}gt^2 = 0.$$

These are simultaneous equations in θ and t .

The first equation gives

$$t = \frac{90}{42 \cos \theta},$$

and using this equation to substitute for t in the second equation gives

$$42 \left(\frac{90}{42 \cos \theta} \right) \sin \theta - \frac{1}{2}g \left(\frac{90}{42 \cos \theta} \right)^2 = 0,$$

which can be simplified to

$$\frac{90 \sin \theta}{\cos \theta} - \frac{90^2 g}{2 \times 42^2 \cos^2 \theta} = 0.$$

Rearranging, we get

$$\frac{90 \sin \theta}{\cos \theta} = \frac{90^2 g}{2 \times 42^2 \cos^2 \theta},$$

so

$$\sin \theta \cos \theta = \frac{90g}{2 \times 42^2},$$

thus (using the hint)

$$\frac{1}{2} \sin 2\theta = \frac{90g}{2 \times 42^2},$$

hence

$$\sin 2\theta = \frac{90 \times 9.8}{42^2},$$

so

$$\sin 2\theta = 0.5.$$

Since $0^\circ < \theta < 90^\circ$, we obtain

$$2\theta = 30^\circ \quad \text{or} \quad 2\theta = 150^\circ,$$

so

$$\theta = 15^\circ \quad \text{or} \quad \theta = 75^\circ.$$

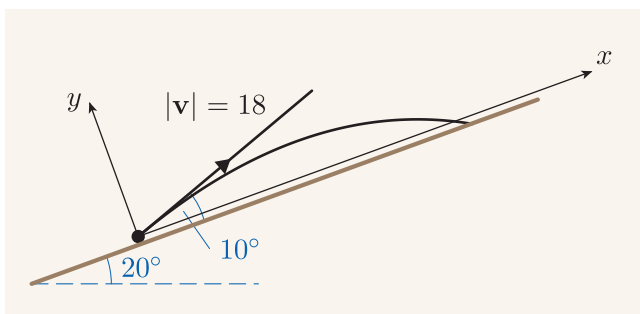
Hence to ensure that the arrow hits the centre of the target, the archer should point the arrow at an angle of 15° above the horizontal. Alternatively, she could point it at an angle of 75° above the horizontal.

Solution to Exercise 28

Model the ball as a particle.

Measure time from the moment when the ball is kicked.

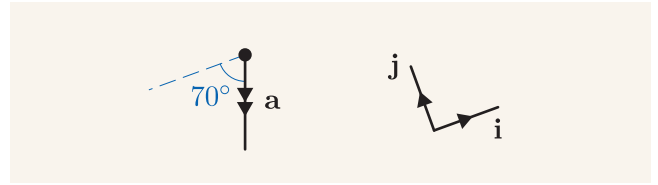
Choose the x -axis to point up the slope and the y -axis to be perpendicular to the slope, with the origin at the point from which the ball is kicked.



We know that at time $t = 0$, the position is $\mathbf{r} = \mathbf{0}$ and the velocity is

$$\mathbf{v} = 18 \cos 10^\circ \mathbf{i} + 18 \sin 10^\circ \mathbf{j}.$$

The acceleration $\mathbf{a}$ is the acceleration due to gravity, so it has magnitude g and points downwards.



Expressing $\mathbf{a}$ in component form gives

$$\mathbf{a} = -g \cos 70^\circ \mathbf{i} - g \sin 70^\circ \mathbf{j}$$

(or equivalently $\mathbf{a} = -g \sin 20^\circ \mathbf{i} - g \cos 20^\circ \mathbf{j}$).

The velocity of the ball is given by

$$\begin{aligned} \mathbf{v} &= \int \mathbf{a} \, dt \\ &= \int (-g \cos 70^\circ \mathbf{i} - g \sin 70^\circ \mathbf{j}) \, dt \\ &= -gt \cos 70^\circ \mathbf{i} - gt \sin 70^\circ \mathbf{j} + \mathbf{c}, \end{aligned}$$

where $\mathbf{c}$ is a vector constant.

At time $t = 0$, the velocity is

$$\mathbf{v} = 18 \cos 10^\circ \mathbf{i} + 18 \sin 10^\circ \mathbf{j}.$$

Substituting these values into the equation above gives

$$\begin{aligned} 18 \cos 10^\circ \mathbf{i} + 18 \sin 10^\circ \mathbf{j} \\ = -g \times 0 \times \cos 70^\circ \mathbf{i} - g \times 0 \times \sin 70^\circ \mathbf{j} + \mathbf{c}, \end{aligned}$$

so

$$\mathbf{c} = 18 \cos 10^\circ \mathbf{i} + 18 \sin 10^\circ \mathbf{j}.$$

This gives

$$\begin{aligned} \mathbf{v} &= -gt \cos 70^\circ \mathbf{i} - gt \sin 70^\circ \mathbf{j} \\ &\quad + 18 \cos 10^\circ \mathbf{i} + 18 \sin 10^\circ \mathbf{j} \\ &= (18 \cos 10^\circ - gt \cos 70^\circ) \mathbf{i} \\ &\quad + (18 \sin 10^\circ - gt \sin 70^\circ) \mathbf{j}. \end{aligned}$$

The position of the ball is given by

$$\begin{aligned} \mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int ((18 \cos 10^\circ - gt \cos 70^\circ) \mathbf{i} \\ &\quad + (18 \sin 10^\circ - gt \sin 70^\circ) \mathbf{j}) \, dt \\ &= (18t \cos 10^\circ - \frac{1}{2}gt^2 \cos 70^\circ) \mathbf{i} \\ &\quad + (18t \sin 10^\circ - \frac{1}{2}gt^2 \sin 70^\circ) \mathbf{j} + \mathbf{d}, \end{aligned}$$

where $\mathbf{d}$ is a vector constant.

Exercise Booklet 10

At time $t = 0$, the position is $\mathbf{r} = \mathbf{0}$.

Substituting these values into the equation above gives

$$\begin{aligned} \mathbf{0} &= (18 \times 0 \times \cos 10^\circ - \frac{1}{2}g \times 0^2 \times \cos 70^\circ) \mathbf{i} \\ &\quad + (18 \times 0 \times \sin 10^\circ - \frac{1}{2}g \times 0^2 \times \sin 70^\circ) \mathbf{j} \\ &\quad + \mathbf{d}, \end{aligned}$$

so $\mathbf{d} = \mathbf{0}$.

This gives

$$\begin{aligned} \mathbf{r} &= (18t \cos 10^\circ - \frac{1}{2}gt^2 \cos 70^\circ) \mathbf{i} \\ &\quad + (18t \sin 10^\circ - \frac{1}{2}gt^2 \sin 70^\circ) \mathbf{j}. \end{aligned}$$

The ball hits the slope when the $\mathbf{j}$ -component of its position is 0, that is, when

$$18t \sin 10^\circ - \frac{1}{2}gt^2 \sin 70^\circ = 0.$$

Factorising this equation gives

$$t(18 \sin 10^\circ - \frac{1}{2}gt \sin 70^\circ) = 0.$$

This equation has two solutions, namely $t = 0$ (which is the moment when the ball is kicked) and the solution t given by

$$18 \sin 10^\circ - \frac{1}{2}gt \sin 70^\circ = 0.$$

Rearranging this equation gives

$$t = \frac{36 \sin 10^\circ}{g \sin 70^\circ}.$$

So this value of t is the time when the ball hits the slope.

The distance travelled up the slope is the $\mathbf{i}$ -component of $\mathbf{r}$, so to find how far up the slope the ball lands, we substitute the value of t obtained above into the formula for the $\mathbf{i}$ -component of $\mathbf{r}$. This gives

$$\begin{aligned} &18 \left(\frac{36 \sin 10^\circ}{g \sin 70^\circ} \right) \cos 10^\circ \\ &\quad - \frac{1}{2}g \left(\frac{36 \sin 10^\circ}{g \sin 70^\circ} \right)^2 \cos 70^\circ. \end{aligned}$$

Simplifying this expression gives

$$\begin{aligned} &\frac{648 \sin 10^\circ \cos 10^\circ}{g \sin 70^\circ} - \frac{648 \sin^2 10^\circ \cos 70^\circ}{g \sin^2 70^\circ} \\ &= \frac{648 \sin 10^\circ}{g \sin^2 70^\circ} (\sin 70^\circ \cos 10^\circ - \cos 70^\circ \sin 10^\circ) \\ &\quad \left(= \frac{648 \sin 10^\circ}{g \sin^2 70^\circ} \sin(70^\circ - 10^\circ) \right) \\ &= 11.2610 \dots \end{aligned}$$

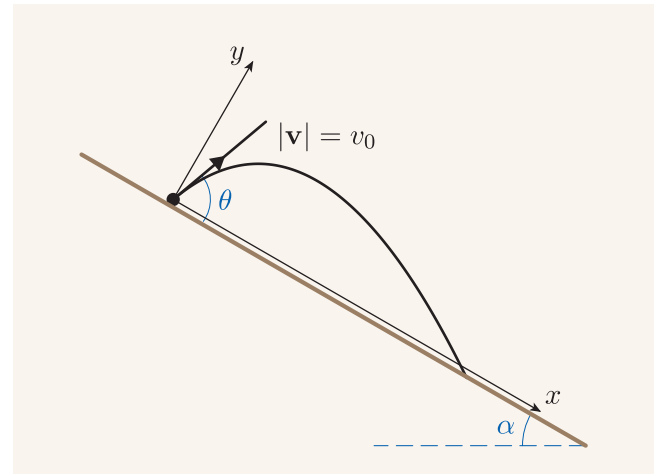
So the ball lands 11 metres (to 2 s.f.) up the slope.

Solution to Exercise 29

(a) Model the ball as a particle.

Measure time from the moment when the ball is thrown.

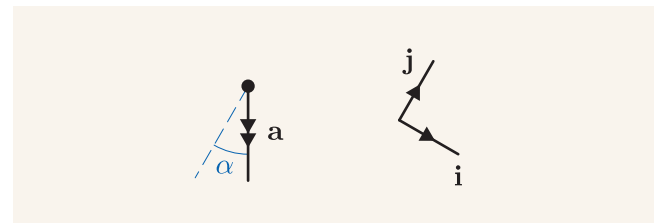
Choose the x -axis to point down the slope and the y -axis to be perpendicular to the slope, with the origin at the point from which the ball is thrown.



We know that at time $t = 0$, the position is $\mathbf{r} = \mathbf{0}$ and the velocity is

$$\mathbf{v} = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j}.$$

The acceleration $\mathbf{a}$ is the acceleration due to gravity, so it has magnitude g and points downwards.



Expressing $\mathbf{a}$ in component form gives

$$\mathbf{a} = g \sin \alpha \mathbf{i} - g \cos \alpha \mathbf{j}.$$

The velocity of the ball is given by

$$\begin{aligned} \mathbf{v} &= \int \mathbf{a} \, dt \\ &= \int (g \sin \alpha \mathbf{i} - g \cos \alpha \mathbf{j}) \, dt \\ &= gt \sin \alpha \mathbf{i} - gt \cos \alpha \mathbf{j} + \mathbf{c}, \end{aligned}$$

where $\mathbf{c}$ is a vector constant.

At time $t = 0$, the velocity is

$$\mathbf{v} = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j}.$$

Substituting these values into the equation above gives

$$\begin{aligned} & v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j} \\ & = g \times 0 \times \sin \alpha \mathbf{i} - g \times 0 \times \cos \alpha \mathbf{j} + \mathbf{c}, \end{aligned}$$

so

$$\mathbf{c} = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j}.$$

This gives

$$\begin{aligned} \mathbf{v} &= gt \sin \alpha \mathbf{i} - gt \cos \alpha \mathbf{j} \\ &\quad + v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j} \\ &= (v_0 \cos \theta + gt \sin \alpha) \mathbf{i} \\ &\quad + (v_0 \sin \theta - gt \cos \alpha) \mathbf{j}. \end{aligned}$$

The position of the ball is given by

$$\begin{aligned} \mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int ((v_0 \cos \theta + gt \sin \alpha) \mathbf{i} \\ &\quad + (v_0 \sin \theta - gt \cos \alpha) \mathbf{j}) \, dt \\ &= (v_0 t \cos \theta + \tfrac{1}{2}gt^2 \sin \alpha) \mathbf{i} \\ &\quad + (v_0 t \sin \theta - \tfrac{1}{2}gt^2 \cos \alpha) \mathbf{j} + \mathbf{d}, \end{aligned}$$

where $\mathbf{d}$ is a vector constant.

At time $t = 0$, the position is $\mathbf{r} = \mathbf{0}$.

Substituting these values into the equation above gives

$$\begin{aligned} \mathbf{0} &= (v_0 \times 0 \times \cos \theta + \tfrac{1}{2}g \times 0^2 \times \sin \alpha) \mathbf{i} \\ &\quad + (v_0 \times 0 \times \sin \theta - \tfrac{1}{2}g \times 0^2 \times \cos \alpha) \mathbf{j} \\ &\quad + \mathbf{d}, \end{aligned}$$

so $\mathbf{d} = \mathbf{0}$. This gives

$$\begin{aligned} \mathbf{r} &= (v_0 t \cos \theta + \tfrac{1}{2}gt^2 \sin \alpha) \mathbf{i} \\ &\quad + (v_0 t \sin \theta - \tfrac{1}{2}gt^2 \cos \alpha) \mathbf{j}. \end{aligned}$$

The ball hits the slope when the $\mathbf{j}$ -component of its position is 0, that is, when

$$v_0 t \sin \theta - \tfrac{1}{2}gt^2 \cos \alpha = 0.$$

Factorising this equation gives

$$t(v_0 \sin \theta - \tfrac{1}{2}gt \cos \alpha) = 0.$$

This equation has two solutions, namely $t = 0$ (which is the moment when the ball is thrown) and the solution t given by

$$v_0 \sin \theta - \tfrac{1}{2}gt \cos \alpha = 0.$$

We can rearrange this equation to give

$$t = \frac{2v_0 \sin \theta}{g \cos \alpha}.$$

So this value of t is the time when the ball lands on the slope.

The distance travelled by the ball down the slope at any time is the $\mathbf{i}$ -component of $\mathbf{r}$, so to find the distance that the ball has travelled down the slope when it lands, we substitute the value of t obtained above into the formula for the $\mathbf{i}$ -component of $\mathbf{r}$. This gives

$$v_0 \left(\frac{2v_0 \sin \theta}{g \cos \alpha} \right) \cos \theta + \tfrac{1}{2}g \left(\frac{2v_0 \sin \theta}{g \cos \alpha} \right)^2 \sin \alpha.$$

Simplifying this expression gives

$$\frac{2v_0^2 \sin \theta \cos \theta}{g \cos \alpha} + \frac{2v_0^2 \sin^2 \theta \sin \alpha}{g \cos^2 \alpha},$$

and combining the fractions gives

$$\frac{2v_0^2 \sin \theta \cos \theta \cos \alpha + 2v_0^2 \sin^2 \theta \sin \alpha}{g \cos^2 \alpha}.$$

Taking out the common factor $2v_0^2 \sin \theta$ in the numerator gives

$$\frac{2v_0^2 \sin \theta (\cos \theta \cos \alpha + \sin \theta \sin \alpha)}{g \cos^2 \alpha}.$$

This is the required expression for the distance travelled by the ball down the slope.

We can use the identity

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

to write the expression above as

$$\frac{2v_0^2 \sin \theta \cos(\theta - \alpha)}{g \cos^2 \alpha},$$

as required.

- (b) Substituting $\alpha = 0$ into the expression above gives

$$\frac{2v_0^2 \sin \theta \cos \theta}{g \cos^2 0},$$

that is,

$$\frac{2v_0^2 \sin \theta \cos \theta}{g}.$$

Using the identity $\sin 2\theta = 2 \sin \theta \cos \theta$, we can write this expression as

$$\frac{v_0^2 \sin 2\theta}{g},$$

as required.